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Report on the doctoral thesis of Elżbieta Krawczyk

The main object studied in this thesis consists of *substitutive sequences* and substitutive systems. These sequences, also called *morphic* (though certain authors define a substitutive sequence to be a non-erasing-morphic sequence), are defined as follows. Start from two finite sets $\mathcal{A}$ and $\mathcal{B}$, also called *alphabets*. Consider the free monoids $\mathcal{A}^*$ and $\mathcal{B}^*$ constructed from these sets. A morphism φ from $\mathcal{A}^* \to \mathcal{B}^*$ is a homomorphism of monoids: it can be defined by a map from $\mathcal{A}$ to $\mathcal{B}^*$, and extended by continuity to the (infinite) sequences on $\mathcal{A}$. An infinite sequence that is fixed point of such a morphism is called *purely substitutive*. The pointwise image of a purely substitutive sequence is called *substitutive*. Such a sequence is called *$q-$automatic* if the morphism has *constant length* q (i.e., if the images of all elements of $\mathcal{A}$ have the same number of letters, namely q).

Now two natural questions come to mind.

(i) Is it possible for a sequence to be both $q-$ and $r-$automatic for integers q and r "sufficiently" different? A theorem due to Cobham (1969) stipulates that only "trivial" sequences (i.e., ultimately periodic sequences) are both $q-$ and $r-$automatic when q and r are multiplicatively independent (i.e. $\log q \,/\log r$ is irrational). The existing proofs of this theorem are intricate in the sense that it is easy to follow each line, but to have a global view of what is happening is painful.

(ii) Suppose that you are given a substitutive sequence, is it q-automatic for some integer $q \geq 2$? (meaning that it can be obtained through another morphism of constant length q). This question was still open, even in the case where the sequence is uniformly recurrent: like many questions with a "simple" statement that are either open or with a complicated proof, it thus sounds very interesting. One of the first non-trivial examples of such a sequence is the Istrail squarefree sequence defined as the fixed point beginning with 1 of the morphism $0 \to 12$, $1 \to 102$, $2 \to 0$: Berstel proved that this sequence can also be obtained as the pointwise image by the reduction modulo 3 of the fixed point beginning with 1 of the uniform morphism $0 \to 12$, $1 \to 13$, $2 \to 20$, $3 \to 21$, hence is 2-automatic. There is a sufficient condition due to Dekking (in 1977-1978) for a purely substitutive sequence to be automatic.

After this introduction that was necessary to understand that the work of Elzbieta Krawczyk is a groundbreaking research, it is time to speak of the work of the candidate. We will describe succinctly some of her results, restricting ourselves to two directions concerning automatic sequences and purely substitutive sequences. The statements below are strongly interesting: they imply all known results on these questions, and, in particular, answer the open question stated above

I. Let $\mathcal{A}$ be an alphabet. Let $\mathcal{U}$ be a subset of $\mathcal{A}^*$, and let $q, r \geq 2$ be two multiplicatively independent integers. The author gives a simple characteristic condition for $\mathcal{U}$ to be the set of common factors of some $q-$automatic and some $r-$automatic sequences. Also for two given $q-$automatic and $r-$automatic sequences, the set of their common factors can be explicitly computed. An immediate corollary of this result is (a stronger version) of Cobham's theorem.

II. E. Krawczyk proves that, given a purely substitutive sequence, if it is also automatic, then the initial morphism is "not far from being of constant length". This condition can be made precise. Furthermore it simplifies in the case where the initial morphism is primitive: then a necessary and sufficient condition for the sequence to be automatic is stated in a way similar to Dekking's result, where the transition matrix of the morphism is replaced with a return substitution associated with the initial substitution.

I am impressed by the fact that Elzbieta Krawczyk needed to master several fields, from combinatorics of words to (discrete) dynamical systems, from ergodic theory to number theory, etc. to obtain her results. Furthermore the thesis is well written and pleasant to read, even though some statements are (necessarily) somehow technical. It also contains a list of more than 100 useful and relevant references. It has already given rise to two joint papers.

It is worth saying that Elzbieta Krawczyk gives some ideas to continue her work further. In particular she mentions the possibility to drop the uniform recurrence property for recognizing that a (purely) substitutive sequence is actually a "hidden" automatic sequence.

I find that this work is of very high quality: it is an excellent doctoral thesis, that fully deserve the mention *cum laude*.

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