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Non-product geometries for particle physics and cosmology

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Oświadczenie

Ja niżej podpisany Arkadiusz Jakub Bochniak doktorant (nr indeksu: 1102259) Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. „**Non-product geometries for particle physics and cosmology**" jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem prof. dr. hab. Andrzeja Sitarza. Pracę napisałem samodzielnie.

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Kraków, dnia 9 lutego 2022,

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Contents

Content of the thesis

The organization of the thesis is as follows. Section 1 is devoted to the introduction to noncommutative geometry together with a brief review of the most important results in the field, chosen by the author as relevant for the rest of the thesis. In section 2 the performed research is motivated and its results are then collected. Each of the subsections 2.1 and 2.2 begin with a brief introduction to the problem. Then each of the articles (or preprints) is reproduced, preceded by an appropriate commentary containing the summary of the obtained results. Each subsection is closed with a brief summary and discussion of possible further research directions in the subject. Finally, in section 3 closing remarks are contained.

This thesis consists of the following articles and preprints, to which the contribution of each author was equal:

1. A. Bochniak and A. Sitarz, *Spectral geometry for the standard model without fermion doubling*, Phys. Rev. D **101**, 075038 (2020), DOI: 10.1103/PhysRevD.101.075038.

2. A. Bochniak, A. Sitarz and P. Zalecki, *Spectral action and the electroweak θ-terms for the Standard Model without fermion doubling*, J. High Energ. Phys. **12**, 142 (2021), DOI: 10.1007/JHEP12(2021)142.

3. A. Bochniak and A. Sitarz, *Stability of Friedmann-Lemaître-Robertson-Walker solutions in doubled geometries*, Phys. Rev. D **103**, 044041 (2021), DOI: 10.1103/PhysRevD.103.044041.

4. A. Bochniak and A. Sitarz, *Spectral interaction between universes*, arXiv: 2201.03839.

5. A. Bochniak *Towards modified bimetric theories within non-product spectral geometry*, arXiv: 2202.03765.

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1 Introduction

1.1 Noncommutative geometry

Geometric structures existing behind physical theories play a crucial role in understanding phenomena predicted by these models. The most recognizable theory formulated in a purely geometric language is Einstein's theory of gravitation - the General Relativity [1]. The attractive idea that fundamental interactions can be encoded by using objects of topological or geometric nature was further extended far beyond the gravitational interactions. In particular, gauge theories can be described in terms of connections and their curvatures defined on certain fiber bundles [2].

1.1.1 The idea

Before making an attempt to *geometrize physics* one has to first answer the question "What a geometry really is?". The common understanding of geometric objects as sets of points, equipped with additional structures that allow us to decide, for example, *how far* are these points from each other, appears to not be fully satisfactory. Fortunately, there is a different, but equivalent, description. Instead of considering the set of points itself, one can concentrate on functions defined on it. More precisely, to a locally compact Hausdorff space $\mathcal{M}$ one can associate the C^*-algebra $C(\mathcal{M})$ of continuous complex-valued functions on $\mathcal{M}$. The Gelfand-Naimark duality theorem [3] assures us that the topology of the space $\mathcal{M}$ can be reconstructed out of the data encoded in $C(\mathcal{M})$. This is the first step to an alternative formulation of the notion of geometry, which can be further easily generalized. Indeed, observe that the C^*-algebra $C(\mathcal{M})$ is commutative, and it is natural to say that if we replace it with a noncommutative C^*-algebra A, it describes *noncommutative topological space*.

The aforementioned duality theorem allows us to store the information about the topology of a space, but a given topological space can be equipped with further structures. To attempt a reformulation of this part, one has to first make use of the Gelfand-Naimark-Segal (GNS) construction [3, 4, 5] that allows us to treat elements of the C^*-algebra A as operators acting on certain Hilbert space $\mathcal{H}$.

The possibility of rewriting topological data in terms of algebraic objects can be further extended, and certain dictionaries can be built. In particular, open (dense) subsets corresponds to (essential) ideals and compactification of a topological space can be translated into the unitalization of a C^*-algebra. For other constructions, we refer the reader to [6, 7].

The first crucial step towards introducing noncommutative analogs of differential and metric structures that one could consider on a given topological space is to replace C^*-algebras A with their dense $*$-subalgebras $\mathcal{A}$. Relaxing the C^*-condition has many advantages as, in particular, it allows for the use of certain differential operators. In classical differential geometry this means that instead of the C^*- algebra $C(\mathcal{M})$ we work with the $*$-algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$. The requirement of smoothness has plenty of implications, and this choice is very natural in the context of differential geometry, where smooth vector fields (sections of certain smooth vector bundles) play an important role.

The fundamental object in differential geometry is the space of one-forms $\Omega^1(\mathcal{M})$, from which the whole differential calculus can be built. This set has a natural structure of $C^\infty(\mathcal{M})$-bimodule, and the exterior derivative d is a derivation $C^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M})$, which is then extendable to $\Omega^\bullet(\mathcal{M})$ by requiring the fulfilment of the Leibniz rule. This observation motivates the definition of a non-commutative first-order differential calculus (FODC) as a pair $(\Omega^1(\mathcal{A}), d)$, where $\Omega^1(\mathcal{A})$ is a bimodule over $\mathcal{A}$, and $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ satisfies $d(ab) = (da) \cdot b + a \cdot (db)$, for all $a, b \in \mathcal{A}$. For a generic algebra $\mathcal{A}$ there usually exist plenty of different FODCs. The simplest choice is the universal one with $\Omega^1_u(\mathcal{A})$ being the kernel of the multiplication map on $\mathcal{A}$, and with the derivation $d_u(a) = a \otimes 1 - 1 \otimes a$, $a \in \mathcal{A}$. The name originates from the fact that for any derivation d on $\mathcal{A}$ with values in a given $\mathcal{A}$-bimodule $\mathcal{E}$ one can find a unique bimodule map $\iota_d : \Omega^1_u(\mathcal{A}) \to \mathcal{E}$ s.t. $d = \iota_d \circ d_u$.

Out of the universal FODC one can construct a differential graded algebra (DGA) $\Omega^\bullet_u(\mathcal{A})$ over $\mathcal{A}$, that is, an $\mathbb{N}$-graded algebra with $\Omega^0_u(\mathcal{A}) = \mathcal{A}$, $\Omega^1_u(\mathcal{A})$ given by the universal FODC, and higher forms (together with the extension of d_u to them) defined in a way that the graded Leibniz rule is satisfied - for details see [7] and [6, Chapt. 8.1].

Having given representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and an (possibly unbounded) operator $F = F^*$ on $\mathcal{H}$ s.t. both $\pi(a)$ and $[F, \pi(a)]$ are bounded for every $a \in \mathcal{A}$, one can show that the assignment

$$\pi_F(a_0 da_1 \dots da_n) = \pi(a_0)[F, \pi(a_1)] \dots [F, \pi(a_n)] \tag{1}$$

defines a representation π_F of $\Omega^\bullet_u(\mathcal{A})$, as DGA, which can be used to construct yet another DGA by diving $\Omega^\bullet_u(\mathcal{A})$ by its ideal $\ker \pi_F + d(\ker \pi_F)$, and if, moreover, $F^2 = 1$, then π_F defines a representation also on the resulting DGA [8].

1.1.2 Fredholm modules and spectral triples

The latter property motivates the notion of an odd (unbounded) Fredholm module over an algebra $\mathcal{A}$. It is defined as a pair (π, F) of an (involutive) representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and an operator $F = F^*$ acting on $\mathcal{H}$ and s.t. $F^2 = 1$ and $[F, \pi(a)]$ is a compact operator for all $a \in \mathcal{A}$ [6, Def. 8.4]. Even Fredholm modules are defined in a similar way but now we assume that $\pi = \pi^0 \oplus \pi^1$ is a representation on a $\mathbb{Z}_2$-graded Hilbert space and F intertwines π^0 and π^1. One may also slightly relax the conditions in the above definitions and demand only that both $\pi(a)(F - F^*)$ and $\pi(a)(F^2 - 1)$ are compact. This leads to the notion of pre-Fredholm modules. Introducing certain equivalence relation between them (see [6, p. 400] for details) we end up with a specific homology theory: the K-homology of $\mathcal{A}$. Pre-Fredholm modules are the so-called K-cycles and their suitable equivalence classes are the K-homology classes. Using the standard Grothendieck's prescription known from K-theory results in two abelian groups, $K^0(\mathcal{A})$ for even modules, and $K^1(\mathcal{A})$ for odd ones.

There is yet another homology theory that can be associated to an algebra $\mathcal{A}$ - the Hochschild homology $HH_\bullet$, defined as a homology of a chain complex with spaces $\mathcal{A}^{\otimes(n+1)}$ and a specific boundary map - for details see [6, Def. 8.14] or [9, Chapt. 5.2]. It encodes the information about the de Rham complex. More precisely, the Hochschild-Kostant-Rosenberg-Connes theorem says that for a compact manifold $\mathcal{M}$, $HH_\bullet(C^\infty(\mathcal{M}))$ is precisely the de Rham complex for $\mathcal{M}$ [6, Chapt. 8.5][1]. In addition to the Hochschild homology one may consider also cyclic homology and cohomology which can be used to define certain invariants of pre-Fredholm modules treated as representants of K-homology classes [12, 13].

The most important examples of pre-Fredholm modules can be constructed out of unbounded self-adjoint operators D on a given Hilbert space $\mathcal{H}$. More precisely, if such D has compact resolvent and $[D, \pi(a)] \in B(\mathcal{H})$, $a \in \mathcal{A}$, then the

[1]See also [10] for the original proof and [11] for the classical algebraic result which was generalized by Connes.

operator $F = D(1+D^2)^{-\frac{1}{2}}$ defines a pre-Fredholm module [6, p. 400],[14]. This observation motivates the definition of a spectral triple. It is defined as a system $(\mathcal{A}, \mathcal{H}, D)$ consisting of an unital $*$-algebra $\mathcal{A}$ (faithfully) represented (by the use of a representation π) on a separable Hilbert space $\mathcal{H}$. D is an essentially self-adjoint operator having compact resolvent, with its domain invariant under the action of $\mathcal{A}$, and s.t. $[D, \pi(a)]$ can be extended to the element in $B(\mathcal{H})$. Spectral triples are also called unbounded K-cycles on $\mathcal{A}$. From now on we identify elements of $\mathcal{A}$ with their images under the representation on $\mathcal{H}$, and omit the symbol of the representation if this will not lead to any confusion.

1.1.3 Grading and reality

The above notion of a spectral triple is, however, very general and, in order to be closer to the usual differential geometry one has to impose further conditions. A *bare* spectral triple can be decorated with other operators defined on the Hilbert space $\mathcal{H}$. One of them is a self-adjoint operator γ that implements a grading on $\mathcal{H}$, i.e. $\gamma^2 = 1$, anticommutes with D and commutes with the elements of $\mathcal{A}$. Such a spectral triple is called even. If such γ is not present, the spectral triple is called odd.

On the $*$-algebra $\mathcal{A}$ we consider an involution, possibly different from the canonical one $a \mapsto a^*$, and we would like to implement also this structure on the Hilbert space $\mathcal{H}$. This can be achieved by choosing an antilinear isometry J on $\mathcal{H}$ s.t. $DJ = \epsilon JD$, $J^2 = \epsilon' \mathrm{id}$ and $J\gamma = \epsilon'' \gamma J$ with $\epsilon, \epsilon', \epsilon'' = \pm 1$.[2] The choice of $(\epsilon, \epsilon', \epsilon'')$ corresponds to the so-called KO-dimension of the spectral triple, which a number from $\mathbb{Z}_8$, and can be understood in terms of KR-cycles - see e.g. [6, Chap. 9.5].

[2]For an odd triple the last relation is omitted.

The above elements are usually supposed to satisfy certain further conditions. For example, by analogy to the Tomita-Takesaki modular theory [15, 16], one may demand that the operator J implements a bimodule structure on $\mathcal{H}$: the left module structure comes from the natural (left) action of $\mathcal{A}$ under its representation on $\mathcal{H}$ via $\mathcal{A} \times \mathcal{H} \ni (a, \psi) \mapsto a\psi \in \mathcal{H}$, while the right module structure is implemented by $\mathcal{H} \times \mathcal{A} \ni (\psi, a) \mapsto Ja^*J^{-1}\psi \in \mathcal{H}$. Notice that this condition is equivalent to demanding that a commutes with Jb^*J^{-1}, for every $a, b \in \mathcal{A}$. It is the so-called zeroth-order condition. A spectral triple that possesses such J is called a real spectral triple.

1.1.4 The first-order condition and Hodge property

Yet other conditions that one can impose on a given spectral triple are related with the operator D. For example, one may require that this operator is within a certain class. In particular, we can demand that it is a noncommutative analogue of differential operator of a given order in derivatives. It turns out that the requirement of being first-order differential operator can be translated into an algebraic language by demanding that certain commutators vanish. More precisely, the so-called first-order condition (for a real spectral triple) is the requirement that $[[D, a], Jb^*J^{-1}] = 0$, for all $a, b \in \mathcal{A}$. This condition can be also generalized into a situation when a triple is not decorated with a real structure J, but then we have to separately assume the existence of a bimodule structure on the Hilbert space $\mathcal{H}$, given by two, possibly independent, left and right representations of $\mathcal{A}$.

The simultaneous fulfillment of order zero and order one conditions is equivalent to the requirement of existing an additional bimodule structure on $\mathcal{H}$. More precisely, let $\Omega^1_D(\mathcal{A})$ denotes the linear span of $\{a[D, b] : a, b \in \mathcal{A}\}$, that is, it is the space of one forms for the spectral triple, with $d = [D, \cdot]$ as a derivation. The Clifford algebra $\mathcal{C}l_D(\mathcal{A})$ for a spectral triple is defined as the complex $*$-subalgebra of $B(\mathcal{H})$ generated by $\mathcal{A}$ and $\Omega^1_D(\mathcal{A})$. One can show that the requirement for $\mathcal{H}$ to be a $\mathcal{C}l_D(\mathcal{A}) - \mathcal{A}$-bimodule is equivalent to the fact that both order zero and order one conditions hold [17, 18].

The situation when the spectral triple is of this type is common. As an example, let $\mathcal{M}$ be a closed four-dimensional Riemannian manifold equipped with a spin structure, and consider the $*$-algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$, and the Hilbert space $L^2(\mathcal{M}, \mathcal{S})$ of square-integrable spinors, being sections of the spinor bundle $\mathcal{S} \to \mathcal{M}$ over $\mathcal{M}$. Under the above assumptions on $\mathcal{M}$, there exists a first-order differential operator $D_\mathcal{M}$ acting on spinors whose action, in local coordinates, can be written as

$$D_\mathcal{M} = i\gamma^\mu(\partial_\mu + \omega_\mu), \tag{2}$$

where ω is the connection on $\mathcal{S}$, we are using the Einstein's summation convention, and gamma matrices satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_4$ with $g_{\mu\nu}$ being the Riemannian metric on $\mathcal{M}$ and I_4 the identity matrix of size four, the dimension of $\mathcal{M}$. The operator $D_\mathcal{M}$ is referrred to as the Dirac operator for the manifold $\mathcal{M}$. Furthermore, since the Clifford algebra generated by gamma matrices is explicitely involved here, we may also make use of the existing γ_5 matrix and the charge conjugation operation, and represent them as operators on $L^2(\mathcal{M}, \mathcal{S})$ to get natural candidates fr a graiding γ and a real structure J, respectively. It is indeed the case, and the system $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, \mathcal{S}), D_\mathcal{M}, \gamma, J)$ forms a real even spectral triple of KO-dimension four with the order one condition satisfied. This triple is usually referred to as the canonical spectral triple for a spin manifold $\mathcal{M}$. Motivated by this example, the operator D for a generic spectral triple is called a Dirac operator (or, by some authors, a Dirac-type operator).

Yet another set of conditions involving Dirac operator is motivated by the study of another spectral triple that arises from classical differential geometry [17, 19]. A real spectral triple is said to satisfy the second-order condition[3] when $J\mathcal{C}l_D(\mathcal{A})J^{-1} \subseteq \mathcal{C}l_D(\mathcal{A})'$, where $\mathcal{C}l_D(\mathcal{A})'$ denotes the commutant of $\mathcal{C}l_D(\mathcal{A})$ in $B(\mathcal{H})$. If, moreover, instead of the inclusion above we have the equality, a spectral triple is said to have the Hodge property. The Hodge condition can be interpreted as a statement that elements of the Hilbert space can be viewed as noncommutative analogues of differential forms [17]. This definition was motivated by the study of the spectral triple constructed for closed oriented Riemannian manifold with the Hilbert space being $L^2\left(\bigwedge_{\mathbb{C}}^{\bullet} T^*\mathcal{M}\right)$ and with the Hogde-de Rham operator as a Dirac operator. The behaviour of the Hodge property under the tensor products was recently examined in [18].

1.1.5 Reconstruction theorem and beyond

We have seen that certain types of spectral triples can arise from manifolds. In the case of spin manifolds, there is a series of interesting features that are possessed by the resulting canonical spectral triple. Furthermore, one can also discuss e.g. the notion of orientability and the Poincaré duality within this framework. The natural question then arises: is it possible to find a set of conditions on a spectral triple that will ensure that this triple comes from the canonical construction? In other words, can the spin manifold (with other structures on it) be reconstructed out of a (decorated) spectral triple? The answer is positive. It turns out that, under certain assumptions on a spectral triple with a commutative algebra $\mathcal{A}$,

[3] See also [20] for a different, but equivalent, formulation of the second-order condition.

it describes the geometry of some smooth compact manifold. This statement was rigorously formulated in the celebrated Connes' reconstruction theorem, first stated as a hypothesis in [21]. The idea of the proof was already presented therein, but some remaining technical steps were, in their final form, made in [22] almost two decades later. Among these requirements, the so-called *Connes' seven axioms* play a central role. One can find a detailed discussion of the roles of the individual axioms in [23], together with a pedagogical introduction to the subject. The original Connes' formulation of these axioms can be found in [22, 24] (see also [25] and [6] for further discussion). Furthermore, as it was shown by A. Connes [26, 27], the knowledge of the Dirac operator is crucial information that could be used to determine the metric structure[4] on $\mathcal{M}$. (For further discussion of the Connes' distance formula see also [28].)

The class of manifolds that is covered by this theorem does not contain several interesting examples. If one would restrict spectral triples only to the ones that satisfy assumptions of the reconstruction theorem, then a large set of objects, which are naturally of geometric nature, will be excluded. In other words, Connes' choice for axioms allows to describe certain geometries, but they cannot be thought of as *the* axioms for the most general geometry - one can work with more general spectral triples. For example, the real structure does not necessarily exist for a generic spectral triple and this corresponds to the distinction between spin and spin$_c$ structures on the manifold. Moreover, there are known natural examples of spectral triples for which the first-order condition is not fulfilled. They originated from the studies of quantum spheres [29, 30, 31, 32].

The aforementioned reconstruction theorem covers only a subclass of Rieman-

[4]For the original ideas of noncommutative spaces see [10].

nian manifolds. One direction for looking for its generalizations is to consider pseudo-Riemannian geometries. This subject was intensively studied in recent years. Nowadays, there is still no analogue of Connes' reconstruction theorem for such commutative geometries and moreover, there is no common agreement on what should be a rigorous definition of pseudo-Riemannian spectral triples, but several proposals, on different levels of exactness, were made. The analysis of classical pseudo-Riemannian Dirac operators in [33] was one of the crucial indications for further directions of search. The certainly not complete list of different proposals for a formulation of noncommutative pseudo-Riemannian geometries contains [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. In [47] yet another formulation generalizing previous ideas from [48] was made.[5]

Yet another generalization can be performed to cover also a class of spaces that are non-compact. Several different proposals for locally compact ones were made, see e.g. [50, 51, 52, 53], but this subject is certainly not closed yet. From the perspective of physical applications, it would be interesting to have a framework that covers both pseudo-Riemannian and non-compact spaces: the Minkowski spacetime, being the fundamental space for the description of field theories for particle physics, is of this type. Furthermore, noncommutative analogs of manifolds with boundaries, as well as with the certain type of singularities, were also discussed [54]. Another examples of nontrivial spectral triples are the ones for noncommutative tori [55] or more general θ-deformations of manifolds [56], Moyal deformation [57] as well as κ-deformations [58, 59].

[5] See also [49] for a brief overview of this approach.

1.1.6 Twisted spectral triples

There is yet another modification of the original approach to noncommutative geometry, introduced by Connes and Moscovici to geometrize certain algebras [60], where the notion of a spectral triple is replaced by its twisted version. We do not intend to discuss here this approach in detail, but let us just mention that the main idea is to replace the usual commutators by their certain modifications by an automorphism ρ of the algebra $\mathcal{A}$. More precisely, the twisted commutator $[D, a]_\rho$ is defined as $Da - \rho(a)D$, $a \in \mathcal{A}$, and the boundedness of them is demanded. Furthermore, other axioms, e.g. the first-order condition, are replaced by their twisted versions [61, 62]. More recently, twisted spectral triples without the twisted first-order condition were also investigated in [63].

One can also try to incorporate some of the twisted conditions into the framework of usual spectral triples. We remark that there exists an interesting class of spectral triples with reality condition modified by a certain twist [64, 65, 66] as well as the ones with multitwisted real structure [67]. The first of them was used to find a relation between twisted spectral triples and the usual ones but with the real structure modified - roughly speaking, it provides a mechanism to *untwist twisted triples*. On the other hand, multitwists allow for the description of yet another type of geometries that contain e.g. the partially rescaled conformal torus [68].

1.2 The spectral action principle

The fact that gravitational interactions can be described purely geometrically can be formulated in a way that emphasises the role of the Hilbert-Einstein action. The cornerstone of the *spectral action principle* is the idea that the form of the physical action can be derived out of spectral data associated with a given manifold. In other words, the spectral triple can be used to produce certain action functional. This groundbreaking idea originated in [69] and then was extensively studied, both from a purely mathematical perspective and possible applications in physical models.[6] We briefly present here the main idea following [23, 71]. We refer to [72] for a rigorous mathematical formulation of spectral action and its properties.

1.2.1 The general construction

For a given spectral triple with the Dirac operator D, and for a parameter called *a cutoff energy scale* $\Lambda > 0$, we consider the operator $D_\Lambda = \frac{|D|}{\Lambda}$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function chosen so that the operator $f(D_\Lambda)$ is of a trace class. The bosonic spectral action is then defined as

$$S_b(D) = \mathrm{Tr} f\left(\frac{|D|}{\Lambda}\right). \tag{3}$$

The meaning of the adjective *bosonic* will become clear later. For most of physical applications the common choice for the function f is by taking it to be the characteristic function of the unit interval $[0,1]$. In this case the spectral action just counts the number of eigenvalues of the Dirac operator which are smaller than the cutoff parameter Λ. It is also possible to take f as an positive even function on the real line and use $\frac{D}{\Lambda}$, which is also a common choice in the literature - see e.g. [9]. Moreover, one can also use $\frac{D^2}{\Lambda^2}$ instead of $\frac{D}{\Lambda}$, but this choice is not always equivalent to the previous ones, e.g. some differences appear when D is not symmetric or the manifolds are not closed.[7][8]

[6] In fact, a certain version of this idea was already present in [70].

[7] See [23] for a discussion of this choice and the problem of regularization.

[8] In the latter case the so-called η-invariants start playing a role. This discussion, however, is beyond the scope of this thesis.

We now briefly summarize here techniques that allow for effective computations of the bosonic spectral action and are the crucial ones for our purposes. In the first part, we will closely follow the presentation in [9]. Our first goal is to have an asymptotic expansion (as $\Lambda \to \infty$) of the bosonic spectral action. In order to achieve this, we have to remind first useful notions and results. Under certain analytical assumptions on a spectral triple (finite summability - the existence of p s.t. $|D|^{-p}$ is of a trace class - and regularity - boundedness of $\delta^k(a)$ and $\delta^k([D,a])$, with $\delta(\cdot) = [|D|, \cdot]$, for all $a \in \mathcal{A}$ and $k \geq 0$), the so-called dimension spectrum can be defined, as a subset of the complex half-plane with non-negative real part, consisting of singularities of analytic functions $\zeta_\ell(z) = \mathrm{Tr}\ell|D|^{-z}$, with ℓ belonging to certain algebra (the one generated by all $\delta^k(a)$ and $\delta^k([D,a])$, $a \in \mathcal{A}$, $k \geq 0$). For details see [9, Def. 5.9]. Using properties of the Mellin transform, assuming that the function f is given by a Laplace-Stieltjes transform of some measure on the positive real half-line and, moreover, that the dimension spectrum is simple, one can show that the following asymptotic expansion of the bosonic spectral action is true [9, Prop. 7.7]:

$$\mathrm{Tr}\, f\left(\frac{D}{\Lambda}\right) = \sum_\ell f_\ell \Lambda^\ell \frac{2}{\Gamma\left(\frac{\ell}{2}\right)} c_{-\frac{1}{2}\ell} + f(0)c_0 + \mathcal{O}\left(\frac{1}{\Lambda}\right), \tag{4}$$

where $f_\ell := \int f(z) z^{\ell-1} dz$, the summation is performed over the dimension spectrum, and c_α are the coefficients of the heat kernel expansion

$$\mathrm{Tr}\, e^{-tD^2} = \sum_\alpha t^\alpha c_\alpha, \tag{5}$$

which exists under the aforementioned assumptions on the spectral triple (see [9, Lemma 7.6]).

1.2.2 Spectral action in terms of Wodzicki residue

Let us now concentrate on the canonical spectral triple for a closed d-dimensional manifold $\mathcal{M}$. The spectral action principle applied to it, in particular, reproduces the Hilbert-Einstein action [24, 73, 74, 75]. This result can be further understood from a slightly different perspective. Consider a finite dimensional vector bundle E over $\mathcal{M}$. One can then consider the algebra $\Psi DO(E)$ of all pseudodifferential operators[9] on it[10]. It was shown by M. Wodzicki [78] that there exists a unique, up to a multiplicative constant, trace on $\Psi DO(E)$, given by

$$\mathrm{Wres}(P) = \frac{1}{(2\pi)^d} \int_{S^*\mathcal{M}} \mathrm{Tr}\left(\sigma^P_{-d}(x,\xi)\right) \mathrm{d}x\, \mathrm{d}\xi, \qquad P \in \Psi DO(E), \tag{6}$$

the Wodzicki residue [79]. Here $\sigma^P_{-d}(x,\xi)$ is the symbol of P of order $-d$, and $S^*\mathcal{M}$ is the associated cosphere bundle.

In case P is a differential operator $\Gamma(E) \to \Gamma(E)$ of order m, given in local coordinates by

$$(P\phi(x))_i = \sum_{j=1}^{\mathrm{rank}(E)} \sum_{|\alpha|=0}^{m} (-i)^{|\alpha|} a^{ij}_\alpha(x) \partial^\alpha_x \phi_j(x), \tag{7}$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$ being a multi-index, $(a^{ij}) \in M_r(\mathbb{C})$ and $\partial^\alpha_x = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, its symbol takes the form $\sum\limits_{|\alpha|=0}^{m} a_\alpha(x)\xi^\alpha$ with $\xi = \xi_\mu dx^\mu$. The principal symbol of P, $\sigma^P_m(x,\xi)$, is defined as the highest term in powers in ξs of this symbol. Operator P is called elliptic if its principal symbol is invertible over $\{(x,\xi) : \xi \neq 0\}$. In was argued in [74], and then revisited in [80], that the coefficients of the heat kernel expansion can be expressed in terms of Wodzicki residua. More precisely, using the properties of the zeta function [76, 79] and the Mellin transform, one can show [80]

[9] See e.g. [76, 77] for the definition and properties of pseudodifferential operators.

[10] In fact, this generality is not necessary for most of our purposes in this thesis, and it is enough to work only with differential operators. Nevertheless, statements here are made in this more general setting.

that the kth coefficient is proportional to the Wodzicki residue of $\frac{d-m}{k}$th power of the inverse of $P = D^2$. This feature was also discussed later e.g. in [81, 82, 83].

The above considerations can be summarized by saying that the bosonic spectral action for the canonical spectral triple associated to a closed four-dimensional Riemannian manifold $\mathcal{M}$ is of the form

$$S_b(D) \sim \Lambda^4 \mathrm{Wres}(D^{-4}) + c\Lambda^2 \mathrm{Wres}(D^{-2}), \tag{8}$$

with some constant c. This is a powerful formula which can be extended to other, more general, spectral triples, e.g. for the noncommutative torus [84].

1.3 Gauge theories and almost-commutative geometries

The idea of using spectral action to derive effective physical Lagrangians, as demonstrated for the Hilbert-Einstein action in the case of General Relativity, can be further extended into Yang-Mills-type theories. The framework that allows for this procedure is the almost-commutative geometry - the one that combines the canonical, commutative, spectral triple for a spin manifold $\mathcal{M}$ with some finite-dimensional one. The resulting object can be roughly thought of as a product $\mathcal{M} \times F$, where F is a finite space [85], and is referred to as an almost-commutative geometry. The idea of applying such product type geometries in particle physics is closely related to the one of Kaluza-Klein models [86, 87, 88], with F playing the role of "an internal space" at every point of the manifold $\mathcal{M}$.[11]

Finite geometries are formally defined as spectral triples with both an algebra $\mathcal{A}_F$ and a Hilbert space $\mathcal{H}_F$ being finite-dimensional. In this case, we are working with matrix algebras, and all operators are just finite matrices. More precisely,

[11] In contrast to the Kaluza-Klein model, the internal space here is finite (i.e. zero dimensional) instead of being a circle.

$\mathcal{A}_F \cong \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$, for some $N > 0$, $\mathcal{H}_F = \bigoplus_{i,j} \mathcal{H}_{ij}$ with $\mathcal{H}_{ij} = \mathbb{C}^{n_i} \oplus \mathbb{C}^{r_{ij}} \otimes \mathbb{C}^{n_j}$ for some $r_{ij} \in \mathbb{N}$. An operator D can be decomposed into components $D_{ij,kl} = P_{ij} D P_{kl}$ with P_{ij} being the projection operator on $\mathcal{H}_{ij}$. Only purely algebraic conditions are relevant, and the ones of analytical nature are void. Finite spectral triples are nowadays well-understood [89, 90, 91]. One can express all the algebraic conditions on finite spectral triples in terms of the components $D_{ij,kl}$ of a Dirac operator and similar ones of other operators involved (e.g. grading). They can be also represented graphically by the use of the so-called Krajewski diagrams. Despite their simplicity, finite spectral triples can describe many non-trivial geometries, including fuzzy spaces [92, 93, 94].

In order to rigorously introduce almost-commutative geometries, we have to first define tensor product of spectral triples. On the level of algebra and Hilbert space this is obvious - we take appropriate tensor products. The only subtlety comes from the matching of KO-dimensions. One has to consistently define both the Dirac operator for the product geometry, the grading for the case of even triples, and the real structure if the triples we started with were in addition real. For the detailed discussion, we refer to [95] as well as [96]. For our purposes, we consider here only a specific example of this construction.

An even almost-commutative (spin) geometry (sometimes called *AC-manifold*) [95, 9] is a spectral triple with an algebra $C^\infty(\mathcal{M}, \mathcal{A}_F) \cong C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$ and a Hilbert space $L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_F$. The Dirac operator for this geometry is of the form

$$D_{\mathcal{M}} \otimes \mathbb{I} + \gamma_5 \otimes D_F, \tag{9}$$

where D_F is a Dirac operator from the finite triple. The grading is the tensor product of the canonical grading for the manifold $\mathcal{M}$ and the one from the finite spectral triple. Similarly for the real structure. Methods developed to the description of finite triples turned out to be useful also for a classification of almost-commutative geometries [97, 98, 99, 100, 101, 102].

We will now briefly summarize how the framework of almost-commutative geometries allows for the description of gauge theories. We follow here [95, 9] and we refer the reader there for a detailed pedagogical introduction to this subject.

We start with the observation that $\mathrm{Aut}(C^\infty(\mathcal{M})) \cong \mathrm{Diff}(\mathcal{M})$, so that, by analogy, one can define $\mathrm{Diff}(\mathcal{M} \times F)$ as the automorphism group of $C^\infty(\mathcal{M}, \mathcal{A}_F)$. It turns out that this is not the full symmetry group of the AC-manifolds. There exist other, *inner*, symmetries of $\mathcal{M} \times F$, implemented by unitary transformations $U : \mathcal{H} \to \mathcal{H}$ being defined by an adjoint action of the unitary group $\mathcal{U}(\mathcal{A})$, i.e. $U = \mathrm{Ad}(u) = uJuJ^{-1}$, $u \in \mathcal{U}(\mathcal{A})$, with $J = J_{\mathcal{M}} \otimes J_F$. The only part of the spectral triple that is affected by this transformation is the Dirac operator, $D \mapsto UDU^*$. The gauge group $\mathcal{G}(\mathcal{M} \times F)$ of an almost-commutative geometry is defined as the set of all transformations U of the above form, and the full symmetry group of this spectral triple is then $\mathcal{G}(\mathcal{M} \times F) \rtimes \mathrm{Diff}(\mathcal{M})$ [9, p. 138] (see also [95, Sec. 2.4.4]). One can show that this group is isomorphic with the automorphism group of a certain pricipal fibre bundle [95, 103, 104].

In order to describe the gauge fields in terms of inner fluctuations we first recall the notion of Morita equivalence for unital algebras $\mathcal{A}$ and $\mathcal{B}$. These algebras are called Morita equivalent if there exists a $\mathcal{B}-\mathcal{A}$-bimodule $\mathcal{E}$ and an $\mathcal{A}-\mathcal{B}$-bimodule $\mathcal{F}$ s.t. $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \cong \mathcal{B}$ and $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \cong \mathcal{A}$ (see [9, Def. 6.9] and the discussion therein). For a spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ we consider yet another one with $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ as an algebra, $\mathcal{E} \otimes \mathcal{H} \otimes \mathcal{E}^\circ$ as a Hilbert space[12], and with the Dirac operator, real structure and the grading defined in an appropriate way - see [9, p. 113] for details

[12]Here $\mathcal{E}^\circ$ denotes the conjugate module of $\mathcal{E}$ (treated as a right $\mathcal{A}$-module) having $\mathcal{E}$ as set of elements but with the left $\mathcal{A}$-module structure defined by the right action by adjoints of elements of $\mathcal{A}$.

of this construction. This can be achieved in a way that if the spectral triple we started with was equipped with a hermitian connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ associated to the derivation $d(a) = [D, a]$, then the resulting collection of data also defines a real even spectral triple [9, Thm. 6.16].

Demanding Morita self-equivalence in the above construction, $\mathrm{End}_{\mathcal{A}}(\mathcal{E}) \cong \mathcal{A}$, the hermitian connection $\nabla : \mathcal{A} \to \Omega^1_D(\mathcal{A})$ has to be of the form $d + \omega$ with $\omega = \omega^* \in \Omega^1_D(\mathcal{A})$, and, finally, the Dirac operator we started with gets fluctuated into $D_\omega = D + \omega + \epsilon J \omega J^{-1}$. Taking $\omega = u[D, u^*]$, the unitary equivalence of spectral triples described before turns out to be a special case of this construction [9, Prop. 6.17]. For further discussion of the role of Morita equivalence we refer to [25].

In order to parametrize fluctuations of the Dirac operator it is convenient to consider a bundle $E = \mathcal{S} \otimes (\mathcal{M} \times \mathcal{H}_F)$ and the so-called twisted connection ∇^E on it (for details of this construction see e.g. [95, p. 23] and [9, p. 141]). Then the operator D^2_ω can be written in the form $\Delta^E - F$, where Δ^E is the Laplacian associated to the twisted connection on E, and F is a certain endomorphism of E ([95, Prop. 3.1] and [9, Prop. 8.6]). For an operator of this form the heat kernel expansion is known [76, Sec. 1.7] and, in the case with $\dim \mathcal{M} = 4$, it takes the form (see [25, Sec. 11] and [95, Sec. 3.2])

$$\mathrm{Tr}\, f\left(\frac{D_\omega}{\Lambda}\right) \sim 2f_4 \Lambda^4 a_0(D^2_\omega) + 2f_2 \Lambda^2 a_2(D^2_\omega) + f(0) a_4(D^2_\omega) + \mathcal{O}\left(\frac{1}{\Lambda}\right), \tag{10}$$

where f_j is the jth moment of the function f, and a_k are the so-called Seeley-DeWitt coefficients. We refer to [105] for a detailed discussion of computational methods of these coefficients, and to [76] for rigorous mathematical formulation.

1.4 Spectral Standard Model

The main idea behind Connes' derivation of models describing particle physics [106] can be summarized as follows: by choosing an appropriate finite spectral triple, considering it as a part of an AC-manifold, and computing the corresponding spectral action, one expects to end up with an action functional, from which an effective Lagrangian describing the physical model can be read.

Since unitary elements of the algebra $\mathcal{A}_F$ are related with the gauge group of the model, one of the simplest candidates for the algebra that could result, in the above sense, in the effective Lagrangian for the Standard Model is

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \tag{11}$$

where $\mathbb{H}$ stands for the algebra of quaternions. The finite-dimensional Hilbert space $\mathcal{H}_F$ is chosen so that its dimension is equal to the number of fermionic degrees of freedom (leptons and quarks) in the physical model. Both particles and antiparticles are treated as independent ones. The chirality is also included, and its presence enlarges the size of $\mathcal{H}_F$ by a factor of two. Yet another factor of three follows from the number of generations. Furthermore, the presence of the chiral structure implies the existence of a natural grading on $\mathcal{H}_F$ that separates left-handed particles from right ones. The real structure on the finite part is given by an operator that replaces particle with corresponding antiparticle (and vice versa), composed with the complex conjugation.

In order to proceed with the procedure described in the previous section, one has to represent the algebra $\mathcal{A}_F$ on the Hilbert space $\mathcal{H}_F$. For one generation of particles, an element $(\lambda, q, m) \in \mathcal{A}_F$ is represented as $\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \oplus q$ in the leptonic sector, and for each color in the quark sector. The action on antileptons is the scalar one - multiplication by λ - while on antiquark sector (with color included) the operator $\mathbb{I}_4 \otimes m$ is taken.

The choice of a Dirac operator is more subtle. The *standard* choice is based on the assumption that this operator commutes with the subalgebra $\{(\lambda, \lambda, 0)\}$ - the physical requirement of having massless photon [71]. Then one can argue that the family of possible choices for the Dirac operator D_F has to be reduced to a certain class of them [107, 108]. An explicit form of the usually made choice for this operator can be found e.g. in [9, Chapt. 11]. See also [109] for a detailed discussion of a noncommutative Standard Model. Many historical remarks can be found also in [23, 9, 71].

Despite the fact that the above choice for an algebra seems to be very natural and one may deduce its appearance almost immediately, in reality it is not the case. The natural question is if the axioms of a spectral triple can uniquely determine the form of the triple that describes the Standard Model. This problem was considered in [110, 111], where the authors fixed all the parts of a spectral triple but not an algebra, and tried to determine possible choices that are compatible with the rest of the triple. They argued that the Standard Model can be thought of as *the smallest possible noncommutative space*. It was also demonstrated that the lack of the first-order condition, in the *minimal* case, leads to the algebra $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$, which gives rise to a family of Pati-Salam models [112] - one of the simplest models going beyond the Standard Model - intensively studied in recent years from the perspective of the noncommutative geometry [113, 114, 115, 116, 117, 118]. Spectral triples without the first-order condition and consequences of its lack were analysed from yet another angle in [119]. Even more general geometry can be obtained by further relaxing the condition of commutation between elements of the algebra and the grading. One ends up then with the so-called Grand Symmetry model [111], with $\mathcal{A}_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$.

Yet another way of analyzing the moduli space of spectral triples for the Standard Model is by fixing the algebra, $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, as well as the Hilbert space, and examine possible choices for the remaining elements of the spectral triple. This way of proceeding was present in [19, 17]. Different types of additional axioms can be imposed in order to uniquely fix the form of the Dirac operator. In particular, in [47] we have proposed to take into account the pseudo-Riemannian structure of the finite spectral triple in order to reduce the number of acceptable Dirac operators, and, as a result, eliminate the possibility of the existence of leptoquarks. This idea was later applied also in the case of the Pati-Salam model [118], reducing them to the so-called Left-Right symmetric ones [120].

Most of the aforementioned choices for a spectral triple of the Standard Model were studied not only on the level of finite triples but also within the full almost-commutative framework. The resulting bosonic spectral action, together with the fermionic one[13], $S_f(D_\omega) = \langle \psi, D_\omega \psi \rangle$, reproduces the Euclidean physical Lagrangian of the Standard Model (see [9, Chapt. 11] and [109] for details; this computation was performed in a full glory for the first time in [107]). Notice that both in the Hilbert space and in the Dirac operator, only the information about the fermionic degrees of freedom were directly encoded: the basis of $\mathcal{H}_F$ was chosen in a way that each of its elements corresponds to an elementary fermionic degree of freedom, and entries of the finite Dirac operator contain Yukawa matrices. The bosonic fields appear then as a result of taking all the fluctuations of the product Dirac operator. Since the fluctuations are closely related to the gauge group of the

[13] See e.g. [9, note on p. 131] for the discussion of its precise form and the role of additional *decorations* present in its different definitions. We will come back to this issue in the forthcoming subsection

model, as discussed in Sec. 1.3, it is not surprising at all that the corresponding gauge fields appear as a result of this computation. On the other hand, it is worth stressing that also the Higgs field (with a proper shape of the potential term) originates out of exactly the same procedure. In other words, within this framework both gauge fields and the Higgs boson are put on the same footing. Furthermore, the number of free parameters in the resulting Lagrangian is smaller than in the one usually used in particle physics, and this feature allows e.g. for the computation of a mass of the Higgs boson. This was performed using the renormalization group techniques - for the details see [25, 107, 121] and also [9, Chapt. 12] for a pedagogical introduction to the subject. The numerical value of this mass, resulting from this computation, is close to the one measured experimentally, but not identical, within the statistical error. The existence of this discrepancy was one of the motivations to search for modifications of the spectral triples introduced by Connes. The list of different proposals contains, and is not limited to, models with additional fermions [122], models with the so-called σ-field [123] (responsible also for curing the vacuum instability issue, originally known from the Weinberg-Salam theory [124, 125]), the Grand Symmetry approach [111], or models based on twisted spectral triples [126].

1.4.1 The fermion doubling problem

We now concentrate on the fermionic content of the noncommutative Standard Model. Within the almost-commutative framework, the full Hilbert space is of the product type, $L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_F$. The finite Hilbert space $\mathcal{H}_F$ decomposes into $\mathcal{H}_{F,L} \oplus \mathcal{H}_{F,R} \oplus \mathcal{H}^c_{F,R} \oplus \mathcal{H}^c_{F,L}$, where $\mathcal{H}_{F,L/R}$ corresponds to left/right particles, while the ones with the superscript c correspond to antiparticles. Similar decomposition exists also for the spinorial part, $L^2(\mathcal{M}, \mathcal{S})$. The tensor product structure introduces then redundancy in the degrees of freedom: there are four times more elements of the basis of the full Hilbert space than really needed. Therefore, a method for eliminating the redundant ones has to be developed in order to end up with the physical Lagrangian. This issue was for the first time noticed in [127], and the authors related it to similar behaviour of mirror fermions in chiral gauge theories[14]. This problem was also discussed in [129], where the possible role of the Lorentzian structure was observed. One factor of two in the counting of the degrees of freedom can be eliminated by introducing certain projection [107, 127] and considering

$$\mathcal{H}_+ = \frac{1 + \gamma_5 \otimes \gamma_F}{2} \left(L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_F \right) \tag{12}$$

as the physical Hilbert space of the model[15]. The modified fermionic spectral action is then taken to be of the form

$$S_f(D_\omega) = \frac{1}{2} \langle J\psi, D_\omega \psi \rangle, \qquad \psi \in \mathcal{H}_+. \tag{13}$$

The presence of the real structure J also cures the other factor of two. Following [130], the first type of doubling is called *the mirror doubling*, while the second one is *the charge-conjugation doubling*. The above way of solving the overcounting problem may seem to be quite an ad hoc, but in [130] it was argued that the Lorentz symmetry, together with an appropriately understood Wick rotation, can explain the origin of this procedure.

[14] A follow up of this idea one can find in [128], where the mirror fermions were discussed in more details. We remark that the spectral triple discussed therein can be thought of as a certain example of non-product geometry.

[15] See also [25, Chapt. 16.3] for a further discussion.

1.4.2 Lorentzian formulation

The spectral action formalism requires the Euclidean framework. On the other hand, the physical Standard Model is described in the Lorentzian framework, and one has to then develop a consistent way of going from one formulation to the other. It is usually done on the level of path integrals by using the Wick rotation. This may in principle leads to potential problems, especially when one transforms Euclidean fermionic quantities or topological terms into the Lorentzian ones (and vice versa). Since there is known no consistent definition of the fully pseudo-Riemannian bosonic spectral action, although some of the existing attempts in this direction are promising [131, 132], the Wick rotation is nowadays the most powerful tool. Nevertheless, one can try to take into account the Lorentzian structure on a different level. It is nothing unexpected that symmetries of the Minkowski spacetime may manifest themselves in plenty of places. Instead of considering the fully Lorentzian model one may follow the aforementioned idea. The first consistent formulation of the Lorentzian noncommutative Standard Model was performed by J. Barrett in [35]. The crucial observation was that demanding the existence of the Lorentzian structure for the manifold requires certain changes in the finite part that led to the conclusion that its KO-dimension has to be equal to six[16]. It is remarkable to stress that also in this case the fermion doubling problem was partially solved and it suggests a deeper connection between the Lorentzian structure and the fermion doubling problem. This is indeed the case, as it was shown in [130], where the authors demonstrated how the Wick rotation has to be consistently performed for both the bosonic and fermionic spectral actions, and that this procedure can be used for an explanation of the previously existing solutions of the fermion doubling

[16] Analogous conclusion was made independently also in [108].

problem.

Yet another approach to the Lorentzian Standard Model was recently re-examined in [133], where the pseudo-Riemannian spectral triples play an essential role. We refer to section 1.1 for a review of different approaches to these types of geometries.

2 New directions

From the discussion in the preceding section, it is clear that noncommutative geometry, in its spectral formulation, is a powerful language, which can be used for the description of models in particle physics as well as for the theory of gravitation. All of the physical models considered so far were analysed within the almost-commutative framework. The natural question that arises is if the requirement of working with geometries of this type is really necessary and if certain types of non-product structures can be found in physical models.

The main goal of this thesis is to analyse certain types of non-product geometries from the perspective of their possible applications in particle physics and cosmology. The thesis has a form of a series of three published articles[17], together with a supplemental material (whose content is available in a form of two preprints). Subsection 2.1 is dedicated to non-product type geometries with applications to the Standard Model of particle physics, while in subsection 2.2 certain cosmological models with non-product structures are examined.

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2.1 Non-product geometry for the Standard Model

Despite the fact that the spectral Standard Model was extensively analysed in the past (see section 1.4 for a more detailed discussion), and a lot of its features are well-understood, it is still an active subject of intense studies. One of the reasons is that there are still some unresolved, or only partially resolved, issues that are of interest due to their physical importance. One has to mention several questions related to the computation of the Higgs mass, the purely Lorentzian formulation, and the computation of spectral action. Furthermore, the interplay between the Lorentzian structure and the fermion doubling problem is yet another aspect that requires further investigation. We stress that we do not claim that none aspects of these problems are not understood at all or some possible ways of finding the solutions were not postulated or examined in the past. On contrary, all of them were studied from different perspectives - we refer to sections 1.4.1 and 1.4.2 for a review of these problems. Yet another disclaimer has to be added: the proposed formulation is possibly not the only solution to the problems we are going to discuss.

Our goal here is to find a formulation of the Standard Model of particle physics, within the framework of noncommutative geometry, which does not assume the almost-commutativity of the corresponding spectral triple. Furthermore, we would like to take into account as much of the Lorentzian structure as possible. Since the fully pseudo-Riemannian formalism is not yet developed, we have to in certain places make use of the Euclidean counterpart. Nevertheless, the role of the Lorentzian structure present in the physical theory plays a crucial role in the formulation and properties of our model.

2.1.1 Formulation of the model and its basic predictions

The Lorentzian structure in the physical field-theoretic description of the Standard Model plays an important role. We propose a model based on noncommutative geometry which can encode the content of the Standard Model and possesses its features. The starting point of our analysis is devoted to the fermionic spectral action. Before discussing the Standard Model itself, the structure of the Minkowski spacetime has to be revisited. First of all, the role of the Krein structure is emphasized. For the canonical Dirac operator $i\gamma^\mu\partial_\mu$ we observe that the fermionic action $\int_{\mathcal{M}} \overline{\psi} D\psi$ can be equivalently written in the form $\int_{\mathcal{M}} \psi^\dagger \widetilde{D}\psi$, where $\widetilde{D} = \gamma^0 D$ is called the Krein-shift of D, and relations satisfied by the Lorentzian Dirac operator can be translated into ones for its Krein-shift.

Using a convenient parametrization of the fermionic content of the Standard Model in terms of four-by-four matrices with entries being the Weyl spinors with a fixed chirality, we then proceed with the construction of the spectral triple that could encode this model. The algebra is taken to be exactly the same as in the case of the almost-commutative Standard Model: algebra of $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$-valued functions over $\mathcal{M}$. However, the Hilbert space differs from the one used in that framework - in our case it is formed from the aforementioned four-by-four matrices. We consider separately the left π_L and right π_R representations of the algebra on this Hilbert space[18]. All the operators acting on this space can be, at every point of the Minkowski spacetime, represented as an element of $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$. This convenient parametrization allows e.g. for discussing the form of a Dirac operator. From the physical perspective, the full Lorentzian Dirac

[18]Notice that the real structure is not present at this level, and the bimodule structure is obtained by explicitly defining left and right representations.

operator differs from the one associated to the Minkowski spacetime by a certain finite endomorphism D_F, which, by a Lorentz invariance, is of the form $M_1 \otimes \mathrm{id} \otimes M_2$ for some $M_1, M_2 \in M_4(\mathbb{C})$. In particular, this part of the Dirac operator has to commute with the natural grading. This leads to a conclusion that the Dirac operator which describes the Lorentzian Standard Model is a sum of two parts, each of them having a different commutation rule with the chirality operator. This clearly demonstrates that our construction is beyond the almost-commutative framework. However, this is a mild generalization - we only reduce the Hilbert space and take a more general Dirac operator than the one for product geometries. We stress that our choice of the Hilbert space automatically solves the fermion doubling problem - there is no mirror doubling as well as the charge-conjugation one. The latter is absent because we do not treat, from the very beginning, antiparticles as independent degrees of freedom, in the same way as they are considered in the physical Standard Model.

Having formulated the model, the natural question of the moduli space of D_F operators arises. The main idea is to demand certain conditions not at the level of this operator itself, but rather by using its Krein-shift. By imposing the (appropriately understood) first-order condition, we show that $\widetilde{D}_F$ does not break color symmetry. Furthermore, the spin$_c$ condition, which requires equality between the image of the right representation and the commutant of the Clifford algebra for the left representation, is automatically satisfied. For three generations of particles the Hodge condition, however, imposes additional conditions on the entries of $\widetilde{D}_F$ parametrized by the Yukawa matrices. These conditions can be satisfied, in the case of the physical Standard Model, e.g. when there is no massless neutrino.

The interplay between the spin$_c$ and Hodge conditions turns out to be deeper.

Considering the doubled model, with $\pi_L \oplus \pi_R$ as a representation and the real structure acting on the direct sum of two copies of the Hilbert space as a flip map composed with complex conjugation, we show that if the model we started with satisfied the spin$_c$ condition, then its doubled version has the Hodge property.

Moreover, the lack of real structure can be related to the CP symmetry. For one generation of particles, commutation relation between the Krein-shift of the D_F operator and the real structure implemented as complex conjugation on the finite part requires their masses to be real, however, for three generations, which is the case of physical interest, it implies that both the Pontecorvo-Maki-Nakagawa-Sakata and Cabibbo-Kobayashi-Maskawa mixing matrices do not possesses nontrivial phases, i.e. the CP symmetry is preserved.

Finally, we find an intriguing relation between our formulation and the approach based on twisted spectral triples. Instead of the *twisting by grading*, commonly considered in such framework, its modification, to which we can refer to as *twisting by pseudo-Riemannian structure*, appears. In this formulation, the automorphism that produces the twist is constructed out of the γ^0 operator, which explains the origin of this nomenclature.

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Spectral geometry for the standard model without fermion doubling

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We propose a simple model of noncommutative geometry to describe the structure of the standard model, which satisfies spin$_c$ condition, has no fermion doubling, does not lead to the possibility of color symmetry breaking, and explains the CP violation as the failure of the reality condition for the Dirac operator.

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I. INTRODUCTION

The standard model of particle interactions is certainly one of the most successful and one of the best tested theories about the fundamental constituents of matter and the forces between them. Even though we still have no satisfactory description of the strong interactions in the low-energy regime and there are some puzzles concerning masses and the character of neutrinos as well as there are some experimental signs that could point out to new physics, the standard model appears to be robust and verified. Yet neither the content of the fermion sector, the mixing between the families, nor the fundamentally different character of the Higgs boson from other gauge bosons appears to have a satisfactory geometrical explanation.

One of the few theories that aimed to provide a sound geometrical basis for the structure of the standard model, explaining the appearance of the Higgs and symmetry-breaking potential, was noncommutative geometry (see Refs. [1–3]). It was constructed with the core idea that spaces with points can be replaced with algebras and provided a plausible explanation of the gauge group of the standard model and the particles in its representation as linked to the unitary group of a finite-dimensional algebra. Merged with the Kaluza-Klein idea that the physical spacetime has extra dimensions, the geometry of the finite-dimensional algebra (in the noncommutative sense) gave rise to the Higgs field understood as a connection, and the Higgs symmetry-breaking potential appeared as the usual Yang-Mills term in the action.

The original model, which is based on the construction of a product geometry, with the resulting geometry being the tensor product of a usual "commutative" space with the finite-dimensional noncommutative geometry suffers from two problems. First, in the original formulation, it is Euclidean. Second, the product structure leads to the quadrupling of the degrees of freedom in the classical Lagrangian [4,5]. Moreover, the conditions put on the Dirac operator for the finite geometry are not sufficient to restrict the class of possible operators to the physical one, leaving the possibility for the nonphysical $SU(3)$-breaking geometries [6–9]. Though the latter problems appear to have at least a partial solution [8], we believe that they can be completely avoided if the noncommutative geometry behind the standard model is assumed to be spin$_c$ only.

It is worth noting that, in addition to the aforementioned issues in formulating the noncommutative standard model, there is also one significant problem related to the disagreement in the predicted Higgs mass and its experimental value [3]. Furthermore, there is also an accompanying problem related to the low value of the Higgs, known as the Higgs vacuum instability. Several possible solutions have been proposed to fix these problems, starting from adding new fermions [10,11], introducing an additional scalar field (so-called σ field) [12,13], extending the algebra to the Grand Symmetry models, [14,15] or using twisted spectral triples formulation [16,17]. All of the mentioned extensions are still based on the concept of real spectral triples with the product geometry, and they similarly require cutting down the quadrupled number of the degrees of freedom.

In what follows, we present a spin$_c$ description of the geometry for the standard model, which does not require fermion doubling and satisfies the spin$_c$ duality for spinors provided that the mass matrices and mixing matrices are nondegenerate. The crucial role is then played not by the Lorentzian Dirac operator but rather by its Krein-shift $\tilde{D}$, the product of the Krein space fundamental symmetry β and the Dirac operator D. This operator can be understood as the self-adjoint component of the Krein decomposition of the Lorentzian Dirac operator, $D = \beta\tilde{D}$. Moreover, we link the breaking of the J condition between the real structure and the Dirac operator to the appearance of the CP-symmetry breaking in the standard model.

We have to stress that the approach we take is based on the physical Lagrangian of the standard model and it is aiming to put a geometrical meaning to its form and the major features like the lack of strong symmetry breaking or CP violation. Our interpretation, based on the noncommutative spin$_c$ geometry, explains both phenomena, and we consider it as a strong signal to take this model seriously. At present, we cannot provide a precise quantitative result for the Higgs mass in our approach, which may be compared with the experimental value. The explicit spectral action computations for the model are currently in progress [18], and for a next step, we aim to see whether the extended models (with Grand Symmetry additional fermions or a scalar field) will satisfy the spin$_c$ condition.

II. DIRAC OPERATOR FOR THE STANDARD MODEL

The Dirac operator for the four-dimensional Minkowski space is of the form $D = i\gamma^\mu\partial_\mu$, with the gamma matrices satisfying the relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the standard Minkowski metric of signature $(+,-,-,-)$. We use the conventions of [8], so that γ^0 is self-adjoint and the remaining gamma matrices are anti-self-adjoint.

The Lorentz-invariant fermionic action, which leads to the Dirac equation, is

$$\int_M \bar{\psi}D\psi = \int_M \psi^\dagger\tilde{D}\psi, \tag{1}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ and $\tilde{D} = \gamma^0 D$. The operator, $\tilde{D}$, is a symmetric operator, which we call the Krein shift of the Dirac operator. This follows from the properties of the Lorentzian Dirac operator D, which is Krein self-adjoint [19], $D^\dagger = \gamma^0 D\gamma^0$, where γ^0 is the fundamental symmetry of the Krein space. Written explicitly in the chiral representation, it becomes

$$\tilde{D} = i\begin{pmatrix} \sigma^\mu & 0 \\ 0 & \tilde{\sigma}_\mu \end{pmatrix}\partial_\mu, \tag{2}$$

where σ^μ and $\tilde{\sigma}^\mu$ are the standard and associated Pauli matrices, $\tilde{\sigma}^0 = \sigma^0$, $\tilde{\sigma}^k = -\sigma^k$.

The Lorentzian Dirac operator and the related Lorentzian spectral triple have the standard $\mathbb{Z}_2$ grading γ and the charge conjugation operator given,

$$\gamma = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \mathcal{J} = i\gamma^2 \circ cc = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \circ cc, \tag{3}$$

where cc denotes the usual complex conjugation of spinors. The operators D, γ, $\mathcal{J}$ satisfy the usual commutation relations for the geometry of the signature (1,3),

$$D\gamma = -\gamma D, \quad D\mathcal{J} = \mathcal{J}D, \quad \mathcal{J}^2 = 1, \quad \mathcal{J}\gamma = -\gamma\mathcal{J}, \tag{4}$$

whereas for the Krein-shifted operator, we have

$$\tilde{D}\gamma = \gamma\tilde{D}, \quad \tilde{D}\mathcal{J} = -\mathcal{J}\tilde{D}, \quad \mathcal{J}^2 = 1, \quad \mathcal{J}\gamma = -\gamma\mathcal{J}. \tag{5}$$

The so-far accepted and tested experimentally action for the standard model of fundamental interactions can be viewed as the extension of the action for a single bispinor to a family of particles, with the additional terms in the action arising from a slight modification of the Dirac operator by an endomorphism of the finite-dimensional space of fermions.

Before we discuss this extension and the conditions it satisfies, we recall the notion of Riemannian spectral triples and spin$_c$-spectral triples, which form a bigger class than these arising from generalization of the spin geometry only.

III. RIEMANNIAN AND PSEUDO-RIEMANNIAN SPECTRAL TRIPLES

A Riemannian finite spectral triple [20] built over a finite-dimensional algebra A is a collection of data (A, D, H, π_L, π_R), where π_L is the representation of A on H and π_R is the representation of A^{op} (the opposite algebra to A) on H such that

$$[\pi_L(a), \pi_R(b)] = 0, \tag{6}$$

$$[[D, \pi_L(a)], \pi_R(b)] = 0, \tag{7}$$

for all $a \in A$ and $b \in A^{op}$.

We say that the spectral triple is of spin$_c$ (see Ref. [21] and compare with the classical result [22]) type if

$$(Cl_D(\pi_L(A))' = \pi_R(A) \tag{8}$$

or of Hodge type if

$$(Cl_D(\pi_L(A))' = Cl_D(\pi_R(A)). \tag{9}$$

By the generalized Clifford algebra $Cl_D(\pi_L(A))$ [and similarly $Cl_D(\pi_R(A))$], we understand the algebra generated by $\pi_L(a)$ and $[D, \pi_L(b)]$ for all $a, b \in A$.

Of course, genuine Riemannian geometries require further assumption that the operator D has a compact resolvent. In the case of Lorentzian or, more generally, pseudo-Riemannian geometries, we might follow the path of Ref. [19], extending the definition of Lorentzian real spectral triples to Lorentzian spin$_c$ geometries.

IV. FERMIONS AND THE ALGEBRA OF THE STANDARD MODEL

Let us recall a convenient parametrization of the particle content in the one-generation standard model [21],

$$\Psi = \begin{pmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{pmatrix} \in M_4(H_W), \qquad (10)$$

where each of the entries is the Weyl spinor over the Minkowski space with a fixed chirality. For the algebra $\mathcal{A}$, we take the algebra of functions over the Minkowski space, valued in $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, and chose the two representations of the algebra

$$\pi_L(\lambda,q,m)\Psi = \begin{pmatrix} \lambda & & \\ & \bar{\lambda} & \\ & & q \end{pmatrix}\Psi, \quad \pi_R(\lambda,q,m)\Psi = \Psi\begin{pmatrix} \lambda & \\ & m^T \end{pmatrix},$$

where λ, q, and m are complex, quaternion, and $M_3(\mathbb{C})$-valued functions, respectively. The representation π_L acts by multiplying Ψ from the left, whereas π_R acts by multiplying Ψ from the right. This is the reason that we transpose m so that π_R is indeed a representation. Observe that, since left and right multiplication commute, then $[\pi_L(a), \pi_R(b)] = 0$ for all $a, b \in \mathcal{A}$; i.e., the zero-order condition is satisfied. Because of the simplicity of the notation at every point of the Minkowski space, we can encode any linear operator on the space of particles as an operator in $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$, where the first and the last matrices act by multiplication from the left and from the right and the middle $M_2(\mathbb{C})$ matrix acts on the components of the Weyl spinor.

The full Lorentzian Dirac operator of the standard model is, in this notation, of the form

$$D_{\mathrm{SM}}\Psi = \underbrace{\begin{pmatrix} & & i\tilde{\sigma}^\mu\partial_\mu & \\ & & & i\tilde{\sigma}^\mu\partial_\mu \\ i\sigma^\mu\partial_\mu & & & \\ & i\sigma^\mu\partial_\mu & & \end{pmatrix}}_{D}\Psi + D_F\Psi, \qquad (11)$$

where D_F is a finite endomorphism of the Hilbert space $M_4(H_W)$.

First of all, observe that the spatial part D is covariant under the Lorentz transformations so that the Lagrange density (1) is invariant. Indeed, using the $SL(2,\mathbb{C})$ representation of the Lorentz group with an appropriate transformation of the Weyl spinors, it is obvious that D transforms covariantly. On the other hand, D_F will transform covariantly, so the full fermionic action will remain invariant under Lorentz transformations, only if it is an element of $M_4(\mathbb{C}) \otimes \mathrm{id} \otimes M_4(\mathbb{C})$, so it is a scalar from the point of view of Lorentz transformations.

At this point, it is the Lorentz invariance and the requirement that D_F behaves like a scalar under Lorentz transformations that fixes D_F to commute with the chirality Γ, which, in fact, can be written as an element of the algebra of the standard model, $\Gamma = \pi_L(1,-1,1)$. In the end, we have the genuine Lorentzian Dirac operator D that anticommutes with Γ and the finite part of the full Dirac operator, D_F, commuting with Γ, whereas the Krein-shifted parts have the opposite behavior.

Next, we find sufficient conditions for the Krein-shifted operator $\widetilde{D_{\mathrm{SM}}}$ to satisfy the first-order condition for the given algebra and the chosen representation. First, observe that $\tilde{D}$ alone obviously satisfies the order-one condition and therefore we need to check only $\widetilde{D_F}$. Suppose then that

$$[[\widetilde{D_F}, \pi_L(a)], \pi_R(b)] = 0,$$

for all $a, b \in \mathcal{A}$. As any element in $\pi_L(\mathcal{A})$ commutes with $\pi_R(\mathcal{A})$, it suffices to find all $\tilde{D_F}$ that are self-adjoint, commute with the elements from $\pi_R(\mathcal{A})$, and anticommute with Γ. It is easy to see that such operators are restricted to

$$\widetilde{D_F} = \underbrace{\begin{pmatrix} & M_l \\ M_l^\dagger & \end{pmatrix}}_{D_l} \otimes e_{11} + \underbrace{\begin{pmatrix} & M_q \\ M_q^\dagger & \end{pmatrix}}_{D_q} \otimes (1_4 - e_{11}), \qquad (12)$$

where $M_l, M_q \in M_2(\mathbb{C})$.

A. spin$_c$ condition

The Krein-shifted Dirac operator satisfies first-order condition, yet it still may not provide the spin$_c$ spectral geometry. We shall look for necessary and sufficient conditions that the commutant of the (complexified) Clifford algebra, $Cl_D(\pi_L(\mathcal{A}))$, generated by $\pi_L(\mathcal{A})$ and $[\widetilde{D_{\mathrm{SM}}}, \pi_L(\mathcal{A})]$ is $\pi_R(\mathcal{A})$. First, observe that all operators in the so-defined $Cl_D(\pi_L(\mathcal{A}))$ are endomorphisms of the space $M_4(H_W)$, which contain a subalgebra generated by the commutators of $\tilde{D}$ with functions $C^\infty_{\mathbb{C}}(M)$. This subalgebra acts on the Weyl spinors pointwise and can be identified with $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$-valued functions on the Minkowski space. The resulting subalgebra of the Clifford algebra acts only on the Weyl-spinorial components, separately in the left and in the right sectors. The commutant of this algebra in the endomorphisms of the Hilbert $M_4(H_W)$ space is then contained in the $M_4(\mathbb{C}) \otimes \mathrm{id} \otimes M_4(\mathbb{C})$ (at each point of the Minkowski space).

Further, consider the subalgebra generated by the commutators of $\widetilde{D_F}$ with constant functions in $\mathcal{A}$. It is a subalgebra of $M_4(\mathbb{C}) \otimes \mathrm{id} \otimes (\mathbb{C} \oplus \mathbb{C}^{(3)})$-valued constant functions over the Minkowski space, and it is easy to see that both subalgebras generate the full Clifford algebra. Therefore, the common commutant of both parts will be the commutant of the full Clifford algebra.

From the decomposition, it is easy to see that the commutant of the second part is the functions in

$\mathrm{id} \otimes M_2(\mathbb{C}) \otimes (\mathbb{C} \oplus M_3(\mathbb{C}))$ and therefore the common parts are functions valued in $\mathrm{id} \otimes \mathrm{id} \otimes (\mathbb{C} \oplus M_3(\mathbb{C}))$, which indeed is the algebra $\pi_R(\mathcal{A})$.

B. Three generations

Let us consider three families of leptons and quarks, that is, the Hilbert space $M_4(H_W) \otimes \mathbb{C}^3$ with the diagonal representation of the algebra. The only difference from the previous section is that the matrices M_l and M_q are no longer in $M_2(\mathbb{C})$ but in $M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$. As the algebra acts diagonally on the Hilbert space (with respect to the generations), we can again repeat the arguments of Ref. [23] and argue that the spin$_c$ condition will hold if algebras generated by $\pi_L(\mathcal{A})$ and D_l, D_q, respectively, will be full matrix algebras, that is, $(M_4(\mathbb{C}) \otimes \mathrm{id} \otimes \mathrm{id}) \otimes M_3(\mathbb{C})$, independently for the lepton and for quarks.

Since the arguments we have used here are analogous to ones used in the discussion of *full conditions* (Section 4.2.2 in Ref. [23]), we infer the same condition for the Hodge property to be satisfied.

Both M_l and M_q can be diagonalized, yet because of the doublet structure of the left leptons and quarks, the components (up/down) cannot be diagonalized simultaneously. The standard presentation of the mass matrices for the physical standard model is then

$$M_l = \begin{pmatrix} \Upsilon_\nu & 0 \\ 0 & \Upsilon_e \end{pmatrix}, \qquad M_q = \begin{pmatrix} \Upsilon_u & 0 \\ 0 & \Upsilon_d \end{pmatrix},$$

where Υ_e and Υ_u are chosen diagonal with the masses of electron, muon, and tau and the up, charm, and top quarks, respectively, and

$$\Upsilon_\nu = U\widetilde{\Upsilon_\nu}U^\dagger, \qquad \Upsilon_d = V\widetilde{\Upsilon_d}V^\dagger,$$

with diagonal matrices $\widetilde{\Upsilon_\nu}$, $\widetilde{\Upsilon_d}$ providing (Dirac) masses of all neutrinos and down, strange, and bottom quarks, where U is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix and V is the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix.

As was indicated also in Ref. [23], the sufficient condition to fulfill the Hodge property is that for both pairs of matrices $(\Upsilon_\nu, \Upsilon_e)$ and (Υ_e, Υ_d) their eigenvalues are pairwise different. This requirement is satisfied in the case of the physical standard model, provided that there is no massless neutrino (see Sec. 5.3 in Ref. [23]).

C. From spin$_c$ to Hodge condition

Consider for a while the Hilbert space $H_{\mathrm{SM}} = M_4(\mathbb{C})$ with the same left and right representations of the algebra as in the standard model case (the standard model Hilbert space is the tensor product of the above with the space of Weyl fermions). Taken with the Krein-shifted Dirac $\widetilde{D_F}$ operator and $\Gamma = \pi_L(1, -1, 1)$, it is a Euclidean even spectral triple.

Assume now that $\widetilde{D_F}$ is such that the spin$_c$ condition holds. We shall describe now the procedure of the *doubling* of the triple so that the resulting real spectral triple satisfies the Hodge duality and is the *finite spectral triple of the standard model* studied so far as the finite component of the product geometry.

Consider $H^2_{\mathrm{SM}} = H_{\mathrm{SM}} \oplus H_{\mathrm{SM}}$ with the representation $\pi_L \oplus \pi_R$. We define the real structure J as the composition of the Hermitian conjugation with the $\mathbb{Z}_2$ action exchanging the two copies of H_{SM}, so that $J(M_1 \oplus M_2) = M_2^* \oplus M_1^*$.

It is clear that the conjugation by J maps the representation of the algebra A to its commutant. We extend Γ so that the relation $J\Gamma = \Gamma J$ holds and extend the Dirac operator $\widetilde{D_F}$ in the following way:

$$D' = \widetilde{D_F} \oplus 0 + J(\widetilde{D_F} \oplus 0)J^{-1}.$$

Clearly, D' anticommutes with Γ and commutes with J. The Clifford algebra, that is, the algebra generated by $\pi_L \oplus \pi_R$ and the commutators with D', is $Cl_{\widetilde{D_F}}(\pi_L(A)) \oplus \pi_R(A)$. Because before the doubling we had the spin$_c$ condition, it is clear that the commutant of the Clifford algebra contains $\pi_R(A) \oplus Cl_{\widetilde{D_F}}(\pi_L(A))$. It is therefore sufficient to verify that there are no other operators T that map H_{SM} to H_{SM}, which would satisfy that they commute with the representation of $Cl_{\widetilde{D_F}}(\pi_L(A)) \oplus \pi_R(A)$. Identifying the Hilbert space as $\mathbb{C}^{16} \oplus \mathbb{C}^{16}$, we see that the first component of Clifford algebra is $M_4(\mathbb{C}) \oplus M_4(\mathbb{C})^{(3)}$ (acting diagonally on $\mathbb{C}^{16}$; the notation $B^{(n)}$ means that we take n copies of the algebra B), and the second is $\mathbb{C}^{(4)} \oplus M_3(\mathbb{C})^{(4)}$. Since all these algebras are independent of each other, there exists no operator intertwining their actions; hence, the commutant is exactly the one indicated above.

D. Reality and the *CP* violation

Let us take the real structure J acting on the finite part just by the complex conjugation, that is, the real structure implemented on $M_4(H_W)$ simply as $\mathrm{id} \otimes \mathcal{J} \otimes \mathrm{id}$. Of course, it does not implement the usual zero-order condition; however, we still have a milder version of the zero-order condition in the following form:

$$\pi_R(\mathcal{A}) \subset J\pi_L(\mathcal{A})J^{-1}.$$

We have already observed what are the commutation relations between $\tilde{D}$ and $\mathcal{J}$ (and hence J). Next, let us see whether similar commutation relations can be imposed on $\widetilde{D_F}$. As both J^2 as well as the anticommutation with Γ are fixed, we see that by imposing the same KO-dimension (6) for the Euclidean finite spectral triple as for the Lorentzian spatial part we shall have $J\widetilde{D_F} = \widetilde{D_F}J$.

This condition is very mild and means that the mass matrices M_l and M_q have to be real. In case of one generation of particles, it implies that masses of fermions have to be real, which is hardly very restrictive.

Yet the situation changes when we pass to three generations as already discussed above when considering the spin$_c$ condition. Since J acts by complex conjugation, then the requirement $\widetilde{D_F}J = J\widetilde{D_F}$ is still equivalent to the matrices M_l, M_q having only real entries. Using the standard parametrization described above, this leads to the reality of the physical masses. However, since in the case of three generations the matrices Υ_ν, Υ_d are not diagonal, we must ensure that both U and V mixing matrices are real.

If this is the case, then all phases in the standard parametrization of these matrices should vanish, which physically will have the interpretation of the CP symmetry preservation. However, in case of the CKM mixing matrix it implies that the Wolfenstein parameter $\bar{\eta}$ has to vanish, but experimentally, it is known that $\bar{\eta} = 0.355^{+0.012}_{-0.011}$ [24]. The CP-violating phase δ^ν_{CP} in the neutrino sector, originated from the PMNS mixing matrix, was determined to be $\delta^\nu_{\mathrm{CP}}/\pi = 1.38^{+0.52}_{-0.38}$ [24,25], which strongly confirms the CP symmetry breaking. Therefore, the existence of CP violation may be interpreted as a shadow of J-symmetry violation in the nondoubled spectral triple.

E. Twisted (pseudo-Riemannian) spectral triple

We have verified that the Krein-shifted Dirac operator satisfies the order-one condition (7). It appears that this is equivalent to the Lorentzian Dirac operator $D_{\mathrm{ST}} = \beta\widetilde{D_{\mathrm{ST}}}$ satisfying a twisted version of the order-one condition, that is,

$$[[D_{\mathrm{ST}}, \pi_L(a)]_\beta, \pi_R(b)]_\beta = 0, \tag{13}$$

where $[x, y]_\beta = xy - \beta y \beta^{-1} x$ and $\beta = \mathrm{id} \otimes \gamma^0 \otimes \mathrm{id}$. This follows directly from a simple computation, which uses $\beta^2 = \mathrm{id}$:

$$\begin{aligned} 0 &= [[\widetilde{D_{\mathrm{ST}}}, \pi_L(a)], \pi_R(b)] \\ &= \beta D_{\mathrm{ST}}\pi_L(a)\pi_R(b) - \pi_L(a)\beta D_{\mathrm{ST}}\pi_R(b) - \pi_R(b)\beta D_{\mathrm{ST}}\pi_L(a) + \pi_R(b)\pi_L(a)\beta D_{\mathrm{ST}} \\ &= \beta(D_{\mathrm{ST}}\pi_L(a)\pi_R(b) - \beta\pi_L(a)\beta D_{\mathrm{ST}}\pi_R(b) - \beta\pi_R(b)\beta D_{\mathrm{ST}}\pi_L(a) + \beta\pi_R(b)\pi_L(a)\beta D_{\mathrm{ST}}) \\ &= \beta[[D, \pi_L(a)]_\beta, \pi_R(b)]_\beta. \end{aligned}$$

V. CONCLUSIONS

Let us stress that the geometry of the standard model, as discussed above, is not a product of spectral triples. Nevertheless, it has interesting features, which we summarize here with an outlook for the future research directions.

When restricted to the commutative algebra of real-valued functions (and its complexification), we obtain the even Lorentzian spectral triple with a real structure of KO-dimension 6 [compatible with the signature (1,3)] and with the Dirac operator satisfying the order-one condition.

On the other hand, the restriction of the spectral triple to the constant functions over the Minkowski space gives a Euclidean even spectral triple, which fails to be real. The failure of the real structure to satisfy the commutation relation with the (Krein-shifted) finite part of the Dirac operator is tantamount to the appearance of the violation of CP symmetry in the standard model.

Neither of the restrictions satisfies the spin$_c$ condition, as in both cases, we still consider the full Hilbert space. Yet the full spectral triple satisfies the spin$_c$ condition in the following sense: the Clifford algebra generated by the commutators of the Krein-shifted Dirac operator with the representation π_L of the algebra has, as the commutant, the right representation of the algebra π_R.

There are several possible ramifications of the above observations. First is the disappearance of the product structure; yet even if the triple is not a full product, then possibly it can have some structure of a quotient "spectral geometry." It will be interesting to classify all possible covers and all Dirac operators for them. In the presented spectral triple, the family of allowed Dirac operators that satisfy the spin$_c$ condition is much closer to physical reality as it does not include any color symmetry–breaking operator unlike Ref. [7] and, moreover, the conditions are exactly the same as for the Hodge duality. The failure of the finite spectral triple to be real is then a geometric interpretation of the CP-symmetry breaking in the standard model. Finally, the disappearance of the product structure may have deep consequences for the spectral action. We postpone the discussion of possible effects on the physical parameters of the model for the forthcoming work. It will be interesting to compare the resulting Higgs mass (and other parameters) with both the experimental value and ones determined in other approaches. Further comparison with models going beyond the standard model, like the Pati-Salam model [26] (see also Ref. [27] for the link to pseudo-Riemannian structures), is also an interesting direction for future research.

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2.1.2 Bosonic spectral action

In the previous subsection, we have discussed the fermionic part of the spectral action for the model, based upon a non-product type of noncommutative geometry, that we propose as an alternative description of the Standard Model. Since most of the features present in the physical model are already built up in this geometric description, we claim that also the bosonic part of the spectral action should reproduce the right kinetic and interaction terms of the effective Lagrangian.

We start with the discussion of gauge transformations for our model and a convenient parametrization for inner fluctuations of the Dirac operator. Again, we are working with the Lorentzian framework, and the analysis is performed by using the Krein-shift of the full Lorentzian Dirac operator. Since the real structure is not present in our formulation of the model, we have to clearly distinguish between the left and right representation, and both of them have to be taken into account when we compute the fluctuations. We demonstrate that the unitary group of this model is $\left(U(1) \times SU(2) \times U(3)\right)/\mathbb{Z}_2$, and after imposing a unimodularity condition it further reduces to either $\left(U(1) \times SU(2) \times U(3)\right)/\mathbb{Z}_6$, the gauge group of the Standard Model, or differs from it a by a finite factor, depending on the choice for unimodularity we use.

Having described the fluctuations of the Dirac operator it is natural to proceed with the computation of the bosonic spectral action as a next step. However, as discussed in the introductory section, the fully Lorentzian framework for such computation is not developed yet. Therefore we have to use a different strategy to analyse this part of an action.

As a first step, we compute the action for the spatial part of the Krein-shifted Dirac operator restricted to the situation when all gauge fields are static. In other words, we are using a certain type of dimension reduction, described e.g. in [134], and *eliminate the time* from the considerations. The resulting operator is Hermitian, its square is elliptic, so the usual Euclidean formalism for the computation of the spectral action can be used. We demonstrate that the resulting effective Lagrangian agrees with the static spatial part of the physical Standard Model.

This promising result motivates further studies. We then proceed wit the discussion of the full model but now using the Wick rotation implemented on the level of the algebra of Pauli matrices: we replace σ^j by $i\sigma^j$, for $j = 1, 2, 3$, while σ^0 remains unchanged. In this case the heat trace coefficients can be computed for the operator $D_\omega^\dagger D_\omega$, where D_ω is the Wick-rotated Lorentzian Dirac operator. Again, after performing the computation of the spectral action, we compare the resulting Lagrangian with the one for the physical Standard Model as well as with the one obtained from the almost-commutative framework. To do so, we have to first apply the inverse of the Wick rotation to end up within the Lorentzian framework. We find two differences between our model and the others. Firstly, the intriguing topological terms in the electroweak sector appear. Since the original model had the feature of breaking the CP symmetry, their appearance does not seem to be so unexpected. The second difference is in the exact form of the Higgs potential. The numerical value of one of the parameters differs from the one obtained in the almost-commutative framework. Not only its absolute value is different but also the sign is the opposite one. However, the detailed analysis of all coefficients in the model and relations between them shows that this may not be an issue, provided that certain axiom related to the positivity of moments of the function f, used for the computation of the spectral action, can be relaxed. A rigorous mathematical formulation of this problem requires further investigations, possibly within the fully

Lorentzian framework.

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Spectral action and the electroweak θ-terms for the Standard Model without fermion doubling

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Abstract: We compute the leading terms of the spectral action for a noncommutative geometry model that has no fermion doubling. The spectral triple describing it, which is chiral and allows for CP-symmetry breaking, has the Dirac operator that is not of the product type. Using Wick rotation we derive explicitly the Lagrangian of the model from the spectral action for a flat metric, demonstrating the appearance of the topological θ-terms for the electroweak gauge fields.

Keywords: Non-Commutative Geometry, Gauge Symmetry, CP violation, Quark Masses and SM Parameters

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Contents

1 Introduction

The Standard Model of Particle Physics is a powerful theory that gives a precise and effective description of all fundamental forces apart from gravity. Its predictive power and agreement with experimental results guarantee that it needs to remain the backbone of any fundamental theory of particle interactions. Yet, in contrast to General Relativity, which is deeply rooted in the geometry of space-time, the Standard Model only partially can be explained similarly. The structure of gauge theory and the Yang-Mills action signifies that indeed the geometry plays there a significant role. However, the appearance of a Higgs field and the symmetry-breaking quartic potential are not directly implied by classical geometry.

The hint that the Standard Model has a more subtle structure came from noncommutative geometry and the theory of spectral triples. Founded by Alain Connes to solve significant mathematical problems related to the index theorem of Atiyah and Singer (see [1] for a review), the theory is a well-structured non-trivial generalization of classical differential geometry that allows studying not only differentiable manifolds but also discrete spaces, fractals and quantum deformations of spaces from a novel point of view. Interestingly, the tools of noncommutative geometry allowed to construct models that explain the geometry of the Standard Model [2–4] (see also [5] and [6] for detailed discussion) and its extensions [7–11]. Their structure is similar to Kaluza-Klein models yet with a finite noncommutative algebra instead of the additional dimension of space-time. The geometry of the entire enhanced space-time is determined by a Dirac operator that depends on the metric and the gauge connections, and also includes the Higgs field, which plays a role of a connection over the finite noncommutative component. The spectral action then gives the full gravity and Yang-Mills action with the quartic Higgs potential and minimal couplings between the Higgs and the gauge fields [12].

The story of the noncommutative model-building is, however, not yet complete as the most accepted model is in the Euclidean signature and requires additional assumptions to remove the possibility of the SU(3) symmetry breaking [13, 14] as well as an additional projection onto the physical space of fermions (due to the fermion quadrupling in the model) [15–17]. In the analysis of the Lorentzian case with slight modifications of the spectral triple rules we proved that there exists a model without the fermion doubling and with exact colour SU(3) symmetry [18]. Moreover, the non-product Dirac operator satisfied a slightly modified first-order condition which is tantamount to the spin$_c$ one under certain requirements for mass spectra of fermions. The CP-symmetry breaking in the Standard Model was then geometrically explained as the lack of reality symmetry of the finite component of the Dirac operator as witnessed by nonvanishing of the Wolfenstein parameter and the CP-phase in the neutrino sector.

In the paper, we compute the spectral action for the model we presented in [18]. It needs to be stressed that this model is not of the product-type geometry and therefore the computations and results are not automatically identical to those performed in the series of papers computing the spectral action [12, 19]. In addition, as we start with the Lorentzian model we need to perform a Wick rotation to be able to use heat trace techniques [20] or restrict the model to spatial and time-independent (static) components of the fields. To check the consistency of the computations we perform both operations. The new element of the spectral action, apart from slight differences in the relative coefficients, is the appearance of topological theta terms for the gauge fields in the electroweak sector. This is a characteristic new feature of this model, which is inherently chiral, especially that such terms cannot appear in the spectral action of the product geometries.

2 The starting point: fermions and the algebra of the Standard Model

We begin by briefly reviewing the model as described in details in [18, 22]. The particle content in the one-generation Standard Model can be conveniently parametrized in the following form:

$$\Psi = \begin{pmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{pmatrix} \in M_4(H_W). \tag{2.1}$$

Every entry of the above matrix is a Weyl spinor (from H_W) over the Minkowski space $\mathcal{M}^{1,3}$. The algebra $\mathcal{A}$ is taken to consist of (smooth) $\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$-valued functions over $\mathcal{M}^{1,3}$. We choose its left and right real representations:

$$\pi_L(\lambda,q,m)\Psi=\begin{pmatrix}\lambda & & \\ & \bar{\lambda} & \\ & & q\end{pmatrix}\Psi,\qquad \pi_R(\lambda,q,m)\Psi=\Psi\begin{pmatrix}\bar{\lambda} & \\ & m^\dagger\end{pmatrix},$$

where λ, q and m are complex, quaternion and $M_3(\mathbb{C})$-valued functions, respectively. Since left and right multiplications commute, the zeroth-order condition is satisfied, i.e.

$$[\pi_L(a),\pi_R(b)]=0$$

for all $a,b\in\mathcal{A}$. It is convenient to encode local linear operator acting on the particle content of the model, at every point of $\mathcal{M}^{1,3}$, as an element of $M_4(\mathbb{C})\otimes M_2(\mathbb{C})\otimes M_4(\mathbb{C})$, where the first and the last matrices act by multiplication from the left and from the right, respectively, while the middle $M_2(\mathbb{C})$ matrix acts on the components of the Weyl spinor. For the algebra $\mathcal{A}$ this component is, of course, identity matrix.

Using this notation, the full Lorentzian Dirac operator of the Standard Model can be written of the form,

$$D_{\mathrm{SM}}\Psi=\underbrace{\begin{pmatrix} & & i\widetilde{\sigma}^\mu\partial_\mu & \\ & & & i\widetilde{\sigma}^\mu\partial_\mu \\ i\sigma^\mu\partial_\mu & & & \\ & i\sigma^\mu\partial_\mu & & \end{pmatrix}}_{D}\Psi+D_F\Psi, \tag{2.2}$$

where $\sigma^0=\mathbf{1}_2=\widetilde{\sigma}^0$ and $\widetilde{\sigma}^i=-\sigma^i$, the latter being standard Pauli matrices. D_F is a finite endomorphism of the Hilbert space $M_4(H_W)$.

In [18] the Krein-shifted full Dirac operator of the Standard Model, $\widetilde{D_{\mathrm{SM}}}=\beta D_{\mathrm{SM}}$, where

$$\beta=\begin{pmatrix} & \mathbf{1}_2\\ \mathbf{1}_2 & \end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_4, \tag{2.3}$$

was discussed in details. The Krein-shifted manifold component of the Lorentzian Dirac operator $\widetilde{D}$ in the local Cartesian coordinates over $\mathbb{R}^4$, with a flat metric, is

$$\widetilde{D}=\begin{pmatrix}\mathbf{1}_2 & \\ & \mathbf{0}_2\end{pmatrix}\otimes i\sigma^\mu\partial_\mu\otimes\mathbf{1}_4+\begin{pmatrix}\mathbf{0}_2 & \\ & \mathbf{1}_2\end{pmatrix}\otimes i\widetilde{\sigma}^\mu\partial_\mu\otimes\mathbf{1}_4, \tag{2.4}$$

whereas the Krein-shifted discrete part of the Dirac operator is,

$$\widetilde{D_F}=\underbrace{\begin{pmatrix} & M_l\\ M_l^\dagger & \end{pmatrix}}_{D_l}\otimes\mathbf{1}_2\otimes e_{11}+\underbrace{\begin{pmatrix} & M_q\\ M_q^\dagger & \end{pmatrix}}_{D_q}\otimes\mathbf{1}_2\otimes(\mathbf{1}_4-e_{11}), \tag{2.5}$$

where $M_l, M_q\in M_2(\mathbb{C})$.

The Krein-shifted Dirac operator and the algebra were proven to satisfy the generalized order-one condition, that is for all $a, b \in A$,

$$\Big[\pi_R(a), [\widetilde{D_{\mathrm{SM}}}, \pi_L(b)]\Big] = 0, \qquad \Big[\pi_L(a), [\widetilde{D_{\mathrm{SM}}}, \pi_R(b)]\Big] = 0. \tag{2.6}$$

Note that although the usual order-one condition, which is implemented with the real structure, can also be written in this form, the above generalized version extends it to the case of Riemannian manifolds, which are not spin [23].

The Lorentzian spectral triple for the signature $(1, 3)$ has the standard chirality $\mathbb{Z}_2$-grading γ and the charge conjugation operator, $\mathcal{J}$,

$$\gamma = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \qquad \mathcal{J} = i\gamma^2 \circ cc = i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \circ cc, \tag{2.7}$$

where cc denotes the usual complex conjugation of spinors. The construction can be easily generalized for the three families of leptons and quarks by enlarging the Hilbert space diagonally, i.e. by taking $M_4(H_W) \otimes \mathbb{C}^3$ with the diagonal representation of the algebra. The matrices M_l and M_q in (2.5) are no longer in $M_2(\mathbb{C})$ but in $M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$. Its standard presentation for the physical Standard Model is

$$M_l = \begin{pmatrix} \Upsilon_\nu & 0 \\ 0 & \Upsilon_e \end{pmatrix}, \qquad M_q = \begin{pmatrix} \Upsilon_u & 0 \\ 0 & \Upsilon_d \end{pmatrix},$$

where Υ_e and Υ_u are chosen diagonal with the masses of electron, muon, and tau and the up, charm, and top quarks, respectively, and Υ_ν and Υ_d can be diagonalised by the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix (PMNS matrix) U and the Cabibbo-Kobayashi-Maskawa mixing matrix (CKM matrix) V, respectively:

$$\Upsilon_\nu = U\widetilde{\Upsilon_\nu}U^\dagger, \qquad \Upsilon_d = V\widetilde{\Upsilon_d}V^\dagger.$$

The matrices $\widetilde{\Upsilon_\nu}, \widetilde{\Upsilon_d}$ provide (Dirac) masses of all neutrinos and down, strange, and bottom quarks.

As it was demonstrated in [18] the model has interesting algebraic properties, like the Morita duality (which means that the generalized Clifford algebra is a full commutant of the algebra $\mathcal{A}$) provided that both pairs of matrices $(\Upsilon_\nu, \Upsilon_e)$ and (Υ_u, Υ_d) have pairwise different eigenvalues. Furthermore, without referring to additional symmetries or assumptions the model preserves the SU(3) strong symmetry and allows for the natural breaking of the CP-symmetry, which is linked to the non-reality of the mixing matrices. This is, on the level of the algebra of the model, equivalent to the failure of the finite part of the Krein-shifted Dirac operator to be $\mathcal{J}$-real (see [18] for details).

2.1 The gauge transformations and the Higgs

In this section we extend the model by introducing the fluctuations of the Dirac operator, that is a family of operators obtained from $\widetilde{D_{\mathrm{SM}}}$ by perturbing them with one-forms, that originate from commutators with the Dirac operator. Here, both left and right representations will contribute to the gauge transformations and the fluctuations of the Dirac operator.

A gauge transformation of physical fields is a unitary transformation of the form,

$$U_{LR} = \pi_L(U)\pi_R(U), \tag{2.8}$$

for a unitary element U of the algebra $\mathcal{A}$, so that the gauge-transformed Dirac operator becomes:

$$\widetilde{D_{\mathrm{SM}}}^U = \pi_L(U)\pi_R(U)\widetilde{D_{\mathrm{SM}}}\pi_R(U^\dagger)\pi_L(U^\dagger), \tag{2.9}$$

which, after using the order-zero and order-one conditions, can be rewritten as

$$\widetilde{D_{\mathrm{SM}}}^U = \widetilde{D_{\mathrm{SM}}} + \pi_L(U)[\widetilde{D_{\mathrm{SM}}}, \pi_L(U^\dagger)] + \pi_R(U)[\widetilde{D_{\mathrm{SM}}}, \pi_R(U^\dagger)]. \tag{2.10}$$

The unitary group of the finite algebra is $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(3)$, while the unitaries of the form U_{LR} are elements of the group being a quotient of this group by its diagonal normal subgroup $\mathbb{Z}_2 = \{\pm(1, \mathbf{1}_2, \mathbf{1}_3)\}$.

To parametrize the fluctuations we first start with computing left and right ones separately:

$$\sum_j \pi_L(a_j)[\widetilde{D_{\mathrm{SM}}}, \pi_L(b_j)], \qquad \sum_j \pi_R(a_j)[\widetilde{D_{\mathrm{SM}}}, \pi_R(b_j)], \tag{2.11}$$

where $a_j, b_j \in \mathcal{A} = C^\infty(\mathbb{R}^4, \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$, and the representations π_L and π_R are of the form:

$$\pi_L(a) = \underbrace{\begin{pmatrix} \lambda_a & & \\ & \overline{\lambda_a} & \\ & & q_a \end{pmatrix}}_{a^L} \otimes \mathbf{1}_2 \otimes \mathbf{1}_4, \qquad \pi_R(a) = \mathbf{1}_4 \otimes \mathbf{1}_2 \otimes \underbrace{\begin{pmatrix} \overline{\lambda_a} & \\ & m_a^\dagger \end{pmatrix}}_{a^R} \tag{2.12}$$

where $\lambda_a \in C^\infty(\mathbb{R}^4)$, $q_a \in C^\infty(\mathbb{R}^4, \mathbb{H})$ and $m_a \in C^\infty(\mathbb{R}^4, M_3(\mathbb{C}))$.

We first notice that $[\widetilde{D_F}, \pi_R(b)] = 0$ from the very definition of the representation and the structure of this Dirac operator. Therefore, the only contribution from the right fluctuations can be parametrized as

$$\begin{pmatrix} \mathbf{1}_2 & \\ & \mathbf{0}_2 \end{pmatrix} \otimes \sigma^\mu \otimes \begin{pmatrix} A'_\mu & \\ & G_\mu \end{pmatrix} + \begin{pmatrix} \mathbf{0}_2 & \\ & \mathbf{1}_2 \end{pmatrix} \otimes \widetilde{\sigma}^\mu \otimes \begin{pmatrix} A'_\mu & \\ & G_\mu \end{pmatrix}, \tag{2.13}$$

where $A'_\mu = i \sum_j \overline{\lambda_{a_j}}(\partial_\mu \overline{\lambda_{b_j}})$ and $G_\mu = i \sum_j m_{a_j}^\dagger \left(\partial_\mu m_{b_j}^\dagger\right)$.

Now, we compute the left fluctuations. Starting with the ones following from the $\widetilde{D}$ part of the Dirac operator we immediately get

$$\sum_j \pi_L(a_j)[\widetilde{D}, \pi_L(b_j)] = A_\mu^R \otimes \sigma^\mu \otimes \mathbf{1}_4 + A_\mu^L \otimes \widetilde{\sigma}^\mu \otimes \mathbf{1}_4, \tag{2.14}$$

with

$$A_\mu^R = \begin{pmatrix} A_\mu & & \\ & A'_\mu & \\ & & 0_2 \end{pmatrix}, \quad A_\mu^L = \begin{pmatrix} 0_2 & \\ & W_\mu \end{pmatrix}, \tag{2.15}$$

where $A_\mu = i\sum_j \lambda_{a_j}(\partial_\mu \lambda_{b_j})$, A'_μ is as previously, and $W_\mu = i\sum_j q_{a_j}(\partial_\mu q_{b_j})$. Here A_μ and A'_μ do not describe the single (electromagnetic) U(1) gauge field, but contain also the Z boson counterpart.

Imposing the selfadjointness condition we immediately get $A'_\mu = -A_\mu$ and infer that W_μ is indeed an element of $i\mathfrak{su}(2)$ (as it is enforced to be a real linear combination of Pauli matrices). Similarly, we deduce that G_μ is a U(3) gauge field.

It remains to take into account the contribution from $\widetilde{D_F}$. By a straightforward computation we get

$$\sum_j \pi_L(a_j)[\widetilde{D_F}, \pi_L(b_j)] = \phi^l \otimes 1_2 \otimes e_{11} + \phi^q \otimes 1_2 \otimes (1_4 - e_{11}), \tag{2.16}$$

where

$$\phi^r = \sum_j a_j^L [D_r, b_j^L], \qquad r = l, q. \tag{2.17}$$

Since both matrices M_l and M_q are diagonal, they commute with $\mathrm{diag}(\lambda, \overline{\lambda})$. Denoting by

$$\boldsymbol{\Phi} = \sum_j \begin{pmatrix} \lambda_{a_j} & \\ & \overline{\lambda_{a_j}} \end{pmatrix} \left[q_{b_j} - \begin{pmatrix} \lambda_{b_j} & \\ & \overline{\lambda_{b_j}} \end{pmatrix} \right],$$

we can parametrize those fluctuations, under the assumption of selfadjointness, as:

$$\begin{pmatrix} & M_l \boldsymbol{\Phi} \\ \boldsymbol{\Phi}^\dagger M_l^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes e_{11} + \begin{pmatrix} & M_q \boldsymbol{\Phi} \\ \boldsymbol{\Phi}^\dagger M_q^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes (\mathbf{1}_4 - e_{11}). \tag{2.18}$$

As a result, the fluctuations coming from all the terms can be parametrize as

$$\begin{aligned} \omega = {} & A_\mu e_{11} \otimes \sigma^\mu \otimes (\mathbf{1}_4 - e_{11}) - 2A_\mu e_{22} \otimes \sigma^\mu \otimes e_{11} \\ & - A_\mu e_{22} \otimes \sigma^\mu \otimes (\mathbf{1}_4 - e_{11}) - A_\mu (e_{33} + e_{44}) \otimes \tilde{\sigma}^\mu \otimes e_{11} \\ & + \begin{pmatrix} \mathbf{0}_2 & \\ & W_\mu \end{pmatrix} \otimes \tilde{\sigma}^\mu \otimes \mathbf{1}_4 \\ & + \begin{pmatrix} \mathbf{1}_2 & \\ & \mathbf{0}_2 \end{pmatrix} \otimes \sigma^\mu \otimes \begin{pmatrix} \mathbf{0}_1 & \\ & G_\mu \end{pmatrix} + \begin{pmatrix} \mathbf{0}_2 & \\ & \mathbf{1}_2 \end{pmatrix} \otimes \tilde{\sigma}^\mu \otimes \begin{pmatrix} \mathbf{0}_1 & \\ & G_\mu \end{pmatrix} \\ & + \begin{pmatrix} & M_l \boldsymbol{\Phi} \\ \boldsymbol{\Phi}^\dagger M_l^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes e_{11} + \begin{pmatrix} & M_q \boldsymbol{\Phi} \\ \boldsymbol{\Phi}^\dagger M_q^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes (\mathbf{1}_4 - e_{11}). \end{aligned} \tag{2.19}$$

We denote the fluctuated Dirac operator by $\widetilde{D_{\mathrm{SM}}}^\omega = \widetilde{D_{\mathrm{SM}}} + \omega$.

For a unitary element $U \equiv (u_1, u_2, u_3) \in \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(3)$ the gauge-transformed fluctuated Dirac operator is of the form

$$\widetilde{D_{\mathrm{SM}}}^{\omega^U} = \pi_L(U)\pi_R(U)\widetilde{D_{\mathrm{SM}}}^\omega \pi_R(U^\dagger)\pi_L(U^\dagger). \tag{2.20}$$

The gauge transformation can be therefore implemented by

$$\begin{aligned} \omega \longmapsto \omega^U = {} & \pi_L(U)\pi_R(U)\omega\pi_R(U^\dagger)\pi_L(U^\dagger) \\ & + \pi_L(U)\left[\widetilde{D_{\mathrm{SM}}}, \pi_L(U^\dagger)\right] + \pi_R(U)\left[\widetilde{D_{\mathrm{SM}}}, \pi_R(U^\dagger)\right]. \end{aligned}$$

As a result, the fields $A_\mu, W_\mu, G_\mu, \mathbf{\Phi}$ transform accordingly:

$$\begin{aligned}
A_\mu &\longmapsto A_\mu + u_1(\partial_\mu \overline{u_1}),\\
W_\mu &\longmapsto u_2 W_\mu u_2^\dagger + u_2(\partial_\mu u_2^\dagger),\\
G_\mu &\longmapsto u_3 G_\mu u_3^\dagger + u_3(\partial_\mu u_3^\dagger),\\
\mathbf{1}_2 + \mathbf{\Phi} &\longmapsto \begin{pmatrix} u_1 & \\ & \overline{u_1} \end{pmatrix} (\mathbf{1}_2 + \mathbf{\Phi}) u_2^\dagger.
\end{aligned} \tag{2.21}$$

We remark that in the above derivation the crucial role was played by the fact that the representation of U(1) part of the gauge group commutes with the mass and mixing matrices.

It is known that the gauge group of the Standard Model should contain SU(3) rather than U(3). This can be achieved with a further condition, the unimodularity of the representation, which, however, can be imposed in different ways. In particular, let us observe that the left action of the group is unimodular from the beginning. For the right representation, one could require either the condition that it is unimodular on each fundamental component (chiral lepton and quark) or in the full representation, including all chiral fermions and families. In the first case, the unimodularity condition is equivalent to $u_1 \det u_3 = 1$, whereas in the second case it becomes $(u_1 \det u_3)^{12} = 1$. In the first case, the resulting group is exactly the group of the Standard Model,

$$\left(\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)\right)/\mathbb{Z}_6,$$

whereas in the latter case it is the one described in [6, proposition 11.4], which differs from the gauge group of the Standard Model by a finite factor. Independently, their Lie algebras agree and the finite difference does not affect the structure of the gauge fields. The unimodularity condition on the Lie algebra, instead of the Lie group, level of perturbation means that the trace of a perturbation has to vanish. This condition results in $\mathrm{Tr}(G_\mu) = A_\mu$. We therefore introduce the traceless gauge field $G'_\mu = G_\mu - \frac{1}{3}A_\mu \mathbf{1}_3$ and then in the perturbations we can replace G_μ by $G'_\mu + \frac{1}{3}A_\mu \mathbf{1}_3$, where now G'_μ is assumed to be a SU(3) gauge field. By a slight abuse of notation we will use G_μ instead of G'_μ in the rest of the paper.

3 The spectral action

The spectral action, as defined originally by Chamseddine and Connes [24], makes sense for elliptic operators on Euclidean manifolds and, in the noncommutative generalisation, for spectral triples. To make contact with physics the usual method is to compute the spectral action in the Euclidean setup and then to Wick-rotate it to the Lorentzian signature. Yet this procedure starts directly from the Euclidean formulation of the model, which may not be equivalent to the Lorentzian. On the other hand, it is feasible to start with the genuine Lorentzian spectral triple and then look either for the appropriate spectral action principle (the first steps towards it have already been done in [25]) or use the Wick-rotated Lorentzian operator (so that then we can work with an elliptic operator for which the

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spectral action is computable) and then Wick-rotate the result back to the Lorentzian case. It remains an interesting general question whether both procedures give the same result. Since the results of [25] have not been so far extended to Dirac-type operators, we proceed with the latter procedure, however, to have another check of the result we compute the spectral action of "static and spatial" part of the Dirac operator, which is an elliptic operator (we explain the procedure in subsection 3.2). Finally, let us remark that we perform computations on a flat manifold, which could be taken as a compact 4-torus, however, since all Gilkey-Seeley-DeWitt coefficients are local the results are extendable to the physical action on a Minkowski space.

In the considerations so far (see e.g. [6] and references therein), the spectral action for the Standard Model was computed for the Euclidean model with fermion doubling. Moreover, the assumed bare Dirac operator was of the product type and therefore its square was simply the sum of the squares of the Dirac operators on manifold component and on the discrete component. While this strategy can be justified by the arguments of covariance and geometric character of the action, the relative coefficients and the proportions between them may in general differ, when one considers the Lorentzian and explicitly chiral Dirac operator.

Of course, the best strategy would be to apply a genuine Lorentzian approach (see [25]), however, this appears to be at the moment restricted only to scalar operators and not Dirac-type operators. Therefore we propose two simple, computable methods to obtain an insight into the action of the model, which is motivated by spectral methods.

The first one assumes that we restrict ourselves to the *static and spatial* case, computing the terms of the spectral action for the Krein-shifted Dirac operator that is restricted to the spatial part and with the gauge fields that arise exclusively through static (time-independent) gauge transformations. Such restricted Dirac operator is already a hermitian elliptic operator and one can easily compute the heat trace coefficients of its square. This shall recover the action of the model for the time-independent fields, which is invariant under static gauge transformations. However, one cannot expect that all terms of the action will be present, and their coefficients correct.

The second method takes as the input the *Wick-rotated Lorentzian Dirac* operator D_w. Such operator is elliptic, as its continuous part is just the usual Wick-rotated Dirac operator (with gauge fields) over the flat space-time. However, the discrete part of the operator (which is not Krein-shifted) is alone not hermitian but only normal. Nevertheless one can still compute the heat trace coefficients of $D_w^\dagger D_w$ and then, using the Wick rotation back to the Lorentzian case recover the action functional.

In what follows we assume that we work on a flat compact manifold (torus) so all curvature terms vanish from the spectral action, and we are using a physical parametrisation of fields, described next.

3.1 Spectral action — physical parametrization

Let us now write explicitly the full spectral action in terms of Yukawa parameters and Higgs field in the case of one generation of fermions. Since $\mathbf{\Phi}$ is a quaternionic field it can be parametrize as

$$\mathbf{\Phi} = \begin{pmatrix} \phi_1 & \phi_2 \\ -\overline{\phi_2} & \overline{\phi_1} \end{pmatrix},$$

where ϕ_1, ϕ_2 are two complex fields. Then

$$\Phi_l = M_l(1+\mathbf{\Phi}) = \begin{pmatrix} \Upsilon_\nu H_1 & \Upsilon_\nu H_2 \\ -\Upsilon_e \overline{H_2} & \Upsilon_e \overline{H_1} \end{pmatrix},$$
$$\Phi_q = M_q(1+\mathbf{\Phi}) = \begin{pmatrix} \Upsilon_u H_1 & \Upsilon_u H_2 \\ -\Upsilon_d \overline{H_2} & \Upsilon_d \overline{H_1} \end{pmatrix},$$

where we introduced the Higgs doublet:

$$H \equiv \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 1+\phi_1 \\ \phi_2 \end{pmatrix}.$$

3.2 The spectral action for the full static SM

We consider here the Krein-shifted operator for the static simplified model, computing the coefficients of the spectral action for its spatial part only, which is an elliptic operator. Of course, this will not give the full four-dimensional spectral action, however, we shall at least recover a part of it, valid for the spatial component of all fields under the assumption that they are time-independent. The procedure can be understood as follows. We first restrict the Krein-shifted Dirac operator, together with all its gauge fluctuations, to the 3-dimensional manifold, obtaining an elliptic operator both for the leptonic and quark sectors. Then we perform the standard computation of the Gilkey-Seeley-DeWitt coefficients, using the standard formulae [20], however, we change the dimension-related constants so that they correspond to the four-dimensional case. Equivalently, this can be seen as the spectral action for the product geometry of the spatial Krein-shifted Dirac operator over a three-dimensional Euclidean manifold with a circle of radius 1, for all fluctuations, which do not depend on the coordinate of the circle and resembles the dimensional reduction procedure as presented in [21].

The fluctuated Krein-shifted static Dirac operator for the Standard Model splits into the lepton and the quark sector with the lepton part,

$$\begin{aligned} \widetilde{D_L} = i \begin{pmatrix} \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \partial_j + \begin{pmatrix} & \Phi_l \\ \Phi_l^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \\ + A_j \begin{pmatrix} \sigma^3 - \mathbf{1}_2 & \\ & \mathbf{1}_2 \end{pmatrix} \otimes \sigma^j - \begin{pmatrix} \mathbf{0}_2 & \\ & W_j \end{pmatrix} \otimes \sigma^j, \end{aligned} \tag{3.1}$$

where we have used the identification $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \mathbb{C} \cong M_4(\mathbb{C}) \otimes M_2(\mathbb{C})$ and therefore omitted the third component in the expression above.

This reproduces the correct hypercharges in the leptonic sector: $0, -2, -1, -1$. We remark that for the left particles, the hypercharges are defined as coefficients in terms containing $\widetilde{\sigma}^j$ instead of σ^j.

For the quark sector we have:

$$\begin{aligned}
\widetilde{D_Q} = i \begin{pmatrix} \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \partial_j \otimes \mathbf{1}_3 + \begin{pmatrix} & \Phi_q \\ \Phi_q^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 \\
+ A_j \begin{pmatrix} \sigma^3 + \frac{1}{3}\mathbf{1}_2 & \\ & -\frac{1}{3}\mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \otimes \mathbf{1}_3 \\
- \begin{pmatrix} \mathbf{0}_2 & \\ & W_j \end{pmatrix} \otimes \sigma^j \otimes \mathbf{1}_3 + \begin{pmatrix} \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \otimes G_j.
\end{aligned} \tag{3.2}$$

Again, it gives correct hypercharges for quarks: $\frac{4}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}$. Contributions to the action can be computed separately for the leptonic and the quark sector. The detailed computations are in the appendix A, here we present the final result in the physical parametrization.

3.2.1 The full spectral action

The asymptotic expansion of the spectral action for models on a four dimensional space with a fluctuated Dirac operator D_ω is given by

$$\mathrm{Tr}\left(f\left(\frac{D_\omega}{\Lambda}\right)\right) \sim a_4 f(0) + 2 \sum_{\substack{0\leq k<4 \\ k \text{ even}}} f_{4-k} a_k \frac{\Lambda^{4-k}}{\Gamma\left(\frac{4-k}{2}\right)} + \mathcal{O}(\Lambda^{-1}), \tag{3.3}$$

and reduces simply to

$$\mathrm{Tr}\left(f\left(\frac{D_\omega}{\Lambda}\right)\right) \sim a_4 f(0) + 2a_0 f_4 \Lambda^4 + 2 f_2 \Lambda^2 a_2 + \mathcal{O}(\Lambda^{-1}),$$

where a_k are the so-called Gilkey-Seeley-DeWitt coefficients and can be computed explicitly — see e.g. [6, 20] for the detailed discussion. Here f is a sufficiently regular function (see e.g. [26, chapter 2]) with f_k being its kth moment, and Λ is the cut-off parameter.

We start with the model containing only one generation of particles. In this case we get

$$\begin{aligned}
a_2 &= -\frac{\kappa}{4\pi^2} a \int d^4x |H|^2, \\
a_4 &= \frac{\kappa}{8\pi^2} \int d^4x \left[b|H|^4 + a\mathrm{Tr}|D_j H|^2 + \frac{20}{9}F^2 + \frac{2}{3}\mathrm{Tr}W^2 + \frac{2}{3}\mathrm{Tr}G^2 \right],
\end{aligned}$$

where

$$\begin{aligned}
a &= |\Upsilon_\nu|^2 + |\Upsilon_e|^2 + 3|\Upsilon_u|^2 + 3|\Upsilon_d|^2, \\
b &= |\Upsilon_\nu|^4 + |\Upsilon_e|^4 + 3|\Upsilon_u|^4 + 3|\Upsilon_d|^4,
\end{aligned}$$

and κ is the normalization of the trace.

In case of three generations we have to change the above coefficients into

$$\begin{aligned}
a &= \mathrm{Tr}(\Upsilon_\nu^\dagger \Upsilon_\nu) + \mathrm{Tr}(\Upsilon_e^\dagger \Upsilon_e) + 3\mathrm{Tr}(\Upsilon_u^\dagger \Upsilon_u) + 3\mathrm{Tr}(\Upsilon_d^\dagger \Upsilon_d), \\
b &= \mathrm{Tr}(\Upsilon_\nu^\dagger \Upsilon_\nu)^2 + \mathrm{Tr}(\Upsilon_e^\dagger \Upsilon_e)^2 + 3\mathrm{Tr}(\Upsilon_u^\dagger \Upsilon_u)^2 + 3\mathrm{Tr}(\Upsilon_d^\dagger \Upsilon_d)^2,
\end{aligned}$$

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and we have to multiply the terms with field curvatures by a factor of 3. As a result

$$a_4 = \frac{\kappa}{8\pi^2}\int d^4x \left[b|H|^4 + a\mathrm{Tr}|D_jH|^2 + \frac{20}{3}F^2 + 2\mathrm{Tr}W^2 + 2\mathrm{Tr}G^2\right].$$

Taking $\kappa = 4$ and ignoring the term related to the gravitational constant (i.e. the one $\sim \Lambda^4$) we end up with a model with an effective Lagrangian $\mathcal{L} = \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{gauge}}$, where

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} &= \frac{bf(0)}{2\pi^2}|H|^4 - \frac{2f_2\Lambda^2 a}{\pi^2}|H|^2 + \frac{af(0)}{2\pi^2}\mathrm{Tr}|D_jH|^2,\\ \mathcal{L}_{\text{gauge}} &= \frac{f(0)}{\pi^2}\left(\frac{10}{3}F^2 + \mathrm{Tr}W^2 + \mathrm{Tr}G^2\right).\end{aligned}$$

The above result is in agreement with the one in [6, proposition 11.9], for $c = d = e = 0$ in the notation used therein. Furthermore, notice also that this is consistent (up to an irrelevant global factor) with taking the static part of the Lorentzian Lagrangian for the Standard Model. Indeed, we have

$$\begin{aligned}-F_{\mu\nu}F^{\mu\nu} + |D_\mu H|^2 - V(H) &= -2F_{0j}F^{0j} - F_{jk}F^{jk} + D_0H^\dagger D_0H - D_jH^\dagger D_jH - V(H)\\ &= -F_{jk}F_{jk} - D_jH^\dagger D_jH - V(H) = -\left(F_{jk}F_{jk} + D_jH^\dagger D_jH + V(H)\right).\end{aligned}$$

In particular any prediction related to the mass of the Higgs field remains unchanged.

3.3 Wick rotated model

Let us start with the full Krein-shifted Dirac operator in the leptonic sector,

$$\begin{aligned}\widetilde{D}_L = {} & i\begin{pmatrix}\mathbf{1}_2\otimes\sigma^\mu & \\ & \mathbf{1}_2\otimes\widetilde{\sigma}^\mu\end{pmatrix}\partial_\mu + A_\mu\begin{pmatrix}(\sigma^3-\mathbf{1}_2)\otimes\sigma^\mu & \\ & -\mathbf{1}_2\otimes\widetilde{\sigma}^\mu\end{pmatrix}\\ & + \begin{pmatrix}\mathbf{0}_4 & \\ & W_\mu\otimes\widetilde{\sigma}^\mu\end{pmatrix} + \begin{pmatrix} & \Phi_l\\ \Phi_l^\dagger & \end{pmatrix}\otimes\mathbf{1}_2.\end{aligned}$$

Its Lorentzian counterpart is of the form

$$\begin{aligned}D_L = {} & i\begin{pmatrix} & \mathbf{1}_2\otimes\widetilde{\sigma}^\mu\\ \mathbf{1}_2\otimes\sigma^\mu & \end{pmatrix}\partial_\mu + A_\mu\begin{pmatrix} & -\mathbf{1}_2\otimes\widetilde{\sigma}^\mu\\ (\sigma^3-\mathbf{1}_2)\otimes\sigma^\mu & \end{pmatrix}\\ & + \begin{pmatrix} & W_\mu\otimes\widetilde{\sigma}^\mu\\ \mathbf{0}_4 & \end{pmatrix} + \begin{pmatrix}\Phi_l^\dagger & \\ & \Phi_l\end{pmatrix}\otimes\mathbf{1}_2.\end{aligned}$$

In what follows we perform a Wick rotation on the level of the algebra of Pauli matrices: $\sigma^j \to i\sigma^j$ and consequently $\widetilde{\sigma}^j \to -i\sigma^j$, while the σ^0 remains unchanged. The Wick-rotated Dirac operator in the leptonic sector is then of the form

$$\begin{aligned}D_{L,w} = {} & i\begin{pmatrix} & \mathbf{1}_2\\ \mathbf{1}_2 & \end{pmatrix}\otimes\mathbf{1}_2\partial_0 + i\begin{pmatrix} & -i\mathbf{1}_2\\ i\mathbf{1}_2 & \end{pmatrix}\otimes\sigma^j\partial_j\\ & + A_0\begin{pmatrix} & -\mathbf{1}_2\\ (\sigma^3-\mathbf{1}_2) & \end{pmatrix}\otimes\mathbf{1}_2 + A_j\begin{pmatrix} & i\mathbf{1}_2\\ i(\sigma^3-\mathbf{1}_2) & \end{pmatrix}\otimes\sigma^j\\ & + \begin{pmatrix} & W_0\\ \mathbf{0}_2 & \end{pmatrix}\otimes\mathbf{1}_2 - \begin{pmatrix} & iW_j\\ \mathbf{0}_2 & \end{pmatrix}\otimes\sigma^j + \begin{pmatrix}\Phi_l^\dagger & \\ & \Phi_l\end{pmatrix}\otimes\mathbf{1}_2.\end{aligned} \tag{3.4}$$

For the quark sector we have

$$\begin{aligned}\widetilde{D}_Q = i\begin{pmatrix}\mathbf{1}_2\otimes\sigma^\mu & \\ & \mathbf{1}_2\otimes\widetilde{\sigma}^\mu\end{pmatrix}\otimes\mathbf{1}_3\partial_\mu + A_\mu\begin{pmatrix}\left(\sigma^3+\frac{1}{3}\mathbf{1}_2\right)\otimes\sigma^\mu & \\ & \frac{1}{3}\mathbf{1}_2\otimes\widetilde{\sigma}^\mu\end{pmatrix}\otimes\mathbf{1}_3 \\ + \begin{pmatrix}\mathbf{1}_2\otimes\sigma^\mu & \\ & \mathbf{1}_2\otimes\widetilde{\sigma}^\mu\end{pmatrix}\otimes G_\mu + \begin{pmatrix}\mathbf{0}_2 & \\ & W_\mu\otimes\widetilde{\sigma}^\mu\end{pmatrix}\otimes\mathbf{1}_3 + \begin{pmatrix} & \Phi_q \\ \Phi_q^\dagger & \end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3.\end{aligned} \tag{3.5}$$

Then

$$\begin{aligned}D_Q = i\begin{pmatrix} & \mathbf{1}_2\otimes\widetilde{\sigma}^\mu \\ \mathbf{1}_2\otimes\sigma^\mu & \end{pmatrix}\otimes\mathbf{1}_3\partial_\mu + A_\mu\begin{pmatrix} & \frac{1}{3}\mathbf{1}_2\otimes\widetilde{\sigma}^\mu \\ \left(\sigma^3+\frac{1}{3}\mathbf{1}_2\right)\otimes\sigma^\mu & \end{pmatrix}\otimes\mathbf{1}_3 \\ + \begin{pmatrix} & \mathbf{1}_2\otimes\widetilde{\sigma}^\mu \\ \mathbf{1}_2\otimes\sigma^\mu & \end{pmatrix}\otimes G_\mu + \begin{pmatrix} & W_\mu\otimes\widetilde{\sigma}^\mu \\ \mathbf{0}_4 & \end{pmatrix}\otimes\mathbf{1}_3 + \begin{pmatrix}\Phi_q^\dagger & \\ & \Phi_q\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3,\end{aligned} \tag{3.6}$$

and after performing the Wick rotation we get

$$\begin{aligned}D_{Q,w} = i\begin{pmatrix} & \mathbf{1}_2 \\ \mathbf{1}_2 & \end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3\partial_0 + i\begin{pmatrix} & -i\mathbf{1}_2 \\ i\mathbf{1}_2 & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3\partial_j + \begin{pmatrix}\Phi_q^\dagger & \\ & \Phi_q\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 \\ + A_0\begin{pmatrix} & \frac{1}{3}\mathbf{1}_2 \\ \sigma^3+\frac{1}{3}\mathbf{1}_2 & \end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 + iA_j\begin{pmatrix} & -\frac{1}{3}\mathbf{1}_2 \\ \sigma^3+\frac{1}{3}\mathbf{1}_2 & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3 \\ + \begin{pmatrix} & \mathbf{1}_2 \\ \mathbf{1}_2 & \end{pmatrix}\otimes\mathbf{1}_2\otimes G_0 + \begin{pmatrix} & -\mathbf{1}_2 \\ \mathbf{1}_2 & \end{pmatrix}\otimes\sigma^j\otimes iG_j \\ + \begin{pmatrix} & W_0 \\ \mathbf{0}_2 & \end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 + \begin{pmatrix} & -iW_j \\ \mathbf{0}_2 & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3.\end{aligned} \tag{3.7}$$

Again, all further details of the computations are in the appendix B, and in what follows we present only the final expressions for the Wick-rotated model.

3.3.1 The full spectral action

We consider now the full model with three generations of particles. In this case, using the above results, we get

$$a_2 = \frac{3\kappa}{4\pi^2}a\int d^4x|H|^2, \tag{3.8}$$

and

$$\begin{aligned}a_4 = \frac{\kappa}{8\pi^2}\int d^4x\Big[b|H|^4 - a\mathrm{Tr}|D_\mu H|^2 + \frac{20}{3}F^2 + 2\mathrm{Tr}(W^2) + 2\mathrm{Tr}(G^2) \\ + 12\varepsilon^{jkl}F_{jk}F_{0l} - 6\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l})\Big],\end{aligned} \tag{3.9}$$

where the parameters a and b are as before:

$$\begin{aligned}a &= \mathrm{Tr}(\Upsilon_\nu^\dagger\Upsilon_\nu) + \mathrm{Tr}(\Upsilon_e^\dagger\Upsilon_e) + 3\mathrm{Tr}(\Upsilon_u^\dagger\Upsilon_u) + 3\mathrm{Tr}(\Upsilon_d^\dagger\Upsilon_d), \\ b &= \mathrm{Tr}(\Upsilon_\nu^\dagger\Upsilon_\nu)^2 + \mathrm{Tr}(\Upsilon_e^\dagger\Upsilon_e)^2 + 3\mathrm{Tr}(\Upsilon_u^\dagger\Upsilon_u)^2 + 3\mathrm{Tr}(\Upsilon_d^\dagger\Upsilon_d)^2.\end{aligned}$$

Notice that by construction these parameters are non-negative. Taking $\kappa = 4$ and considering the first terms of the asymptotic expansion (and neglecting the gravitational terms) we end up with the following Lagrangians for gauge fields and the field H:

$$\mathcal{L}_{\text{gauge}} = \frac{f(0)}{\pi^2}\left(\frac{10}{3}F^2 + \mathrm{Tr}(W^2) + \mathrm{Tr}(G^2) + 6\varepsilon^{jkl}F_{jk}F_{0l} - 3\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l})\right), \tag{3.10}$$

$$\mathcal{L}_H = \frac{bf(0)}{2\pi^2}|H|^4 + \frac{6f_2\Lambda^2}{\pi^2}a|H|^2 - \frac{af(0)}{2\pi^2}\mathrm{Tr}|D_\mu H|^2. \tag{3.11}$$

Since the Wick rotation was performed in three spatial directions on the level of Pauli algebra, as described in the discussion preceding eq. (3.4), when going back from the Minkowski signature $(1,3)$ into the Euclidean one we have to change spatial derivatives and coordinates according to $\partial_j \to -i\partial_j$ and $A_j \to -iA_j$, respectively, and in order to preserve the spin structure we have to change the Minkowski structure constants $\varepsilon_{\mathrm{M}}^{jkl}$ into their Euclidean counterparts: $\varepsilon_{\mathrm{E}}^{jkl} = -i\varepsilon_{\mathrm{M}}^{jkl}$. As a result

$$\begin{aligned}-F^{\mathrm{M}}_{\mu\nu}F_{\mathrm{M}}^{\mu\nu} &= -2F^{\mathrm{M}}_{0j}F_{\mathrm{M}}^{0j} - F^{\mathrm{M}}_{jk}F_{\mathrm{M}}^{jk} = 2F^{\mathrm{M}}_{0j}F^{\mathrm{M}}_{0j} - F^{\mathrm{M}}_{jk}F^{\mathrm{M}}_{jk}\\ &\to -2F^{\mathrm{E}}_{0j}F^{\mathrm{E}}_{0j} - F^{\mathrm{E}}_{jk}F^{\mathrm{E}}_{jk} = -F^{\mathrm{E}}_{\mu\nu}F^{\mathrm{E}}_{\mu\nu},\end{aligned}$$

and

$$\begin{aligned}(D_\mu H^\dagger_{\mathrm{M}})(D^\mu H_{\mathrm{M}}) &= (D_0 H^\dagger_{\mathrm{M}})(D_0 H_{\mathrm{M}}) - (D_j H^\dagger_{\mathrm{M}})(D_j H_{\mathrm{M}})\\ &\to (D_0 H^\dagger_{\mathrm{E}})(D_0 H_{\mathrm{E}}) + (D_j H^\dagger_{\mathrm{E}})(D_j H_{\mathrm{E}}) = (D_\mu H^\dagger_{\mathrm{E}})(D_\mu H_{\mathrm{E}}),\end{aligned} \tag{3.12}$$

so that for these terms we have

$$-F^2_{\mathrm{M}} + |D_\mu H_{\mathrm{M}}|^2 - V(H_{\mathrm{M}}) \to -\left(F^2_{\mathrm{E}} - |D_\mu H_{\mathrm{E}}|^2 + V(H_{\mathrm{E}})\right)$$

in a complete agreement with (3.10) and (3.11). The global minus sign (together with the additional $-i$ factor from the measure) is related to the definition of an Euclidean action: $iS_{\mathrm{M}} = -S_{\mathrm{E}}$. Next, let us consider the remaining term:

$$\varepsilon_{\mathrm{M}}^{\mu\nu\rho\sigma}F^{\mathrm{M}}_{\mu\nu}F^{\mathrm{M}}_{\rho\sigma} = 4\varepsilon_{\mathrm{M}}^{jkl}F^{\mathrm{M}}_{0j}F^{\mathrm{M}}_{kl} \to -4\varepsilon_{\mathrm{E}}^{jkl}F^{\mathrm{E}}_{0j}F^{\mathrm{E}}_{jk}.$$

Taking into account the additional global sign from the identification of iS_{M} with $-S_{\mathrm{E}}$, we finally see that the Lorentzian counterpart of $6\varepsilon^{jkl}F_{jk}F_{0l} - 3\varepsilon^{jkl}\mathrm{Tr}\,(W_{jk}W_{0l})$ is

$$\frac{3}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{3}{4}\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,(W_{\mu\nu}W_{\rho\sigma})\,.$$

Therefore, the spectral action for this model contains terms that can be interpreted as the so-called θ-terms in the electroweak sector [27–29]. We remark that from the above derivation of the spectral action not only the presence of such terms is deduced but also the numerical value of the electroweak vacuum angle is fixed by the model. The presence of such terms was linked with the CP-violation [28], especially for the discussion of the baryogenesis process. In contrast to the usual considerations in the physical formulation of the Standard Model, no CP-violating θ-term in the QCD sector is present here. It will be interesting to see what are the physical limitations, e.g. on the energy scales on which such model is valid, from the perspective of the presence of the electroweak θ-terms. The CP-violation was present in this model also on the level of algebra by the lack of the $\mathcal{J}$-symmetry [18].

We remark that the explicit form of the potential $V(H)$ differs from the one in the standard derivation [6], where the coefficient in the quadratic term $|H|^2$ contained $-2f_2$ instead of $6f_2$, which we have in the present model. In the usual formulation, f is assumed to be, besides the others, a non-negative on the positive real half-line, so then f_2 is also non-negative therein. If we would not allow for any relaxation of this principle, our model will not predict the Higgs mechanism, or in other words, the model could be valid only in a sector with the Higgs potential of the form $|H|^4 + b_1|H|^2 + b_2$ with positive b_1, b_2, i.e. the Higgs potential will not possess a non-trivial minimum. On the other hand, having the possibility of using function f which has a negative second moment gives rise to effective action for the Standard Model with the Higgs mechanism implemented in a completely similar manner as in the usual product-like almost-commutative geometry [6]. Since all the derivations were made on the algebraic level we could, by linearity, extend the applicability of the usual methods into the case with f being a difference of two positive functions. However, the discussion of the analytical aspects is required to establish the range of validity of these computational methods — see [26] for some further discussion of these aspects which are beyond the scope of this paper.

Allowing for the negative value of f_2 there is no further difference in the numerical value of the Higgs mass, which can be computed from the derived Lagrangian using the standard tools based on the renormalization group equation [6, 12]. This is because the difference in the numerical value of f_2 in the coefficient for the $|H|^2$ term does not affect any relation between the mass of the W boson, the Higgs vacuum expectation value v and the coupling constant g_2 for the W boson field. Of course, the constant f_2 appears in other, purely gravitational terms, which have been deliberately neglected in these computations. Certainly, the relative sign between the cosmological term and the Einstein-Hilbert term is significant for gravity, however, this depends on another constant f_4, which contributes to the factor in front of the cosmological term. In the usual Einstein-Hilbert action the signs of these two terms are opposite, which is consistent with our results provided that f_4 is positive. The only potential problem for the Euclidean action is its overall positivity, yet this may depend on the overall sign, which depends on the Wick-rotation scheme. We hope that the question of relative signs and spectral action expansion coefficients for the full Lorentzian model will be effectively tackled by extending the results of [25] to Dirac-type operators.

4 Conclusions and outlook

The presented noncommutative geometric model describing the particle interaction appears to be the closest to the observed Standard Model. The derived bosonic spectral action gives all correct terms and, in addition, new, topological θ-terms. While the latter has no significance for the dynamics of the model, it may play a role in the quantum effects for the electroweak sector. These terms are, in principle, not excluded and have been discussed in literature [27–29]. The result signifies also that computing the spectral action for the Wick-rotated Lorentzian Dirac operator is important. Possibly, the next step shall be to compute the genuine Lorentzian spectral action using the tools that are at present available for the

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Laplace-type operators [25]. Furthermore, possible relations of non-product geometries with bundle-like structures over noncommutative manifolds [30] as well to the inclusion of gravity for this non-product geometry (see [31] for a link between nonproduct geometries and gravity) shall also be explored and examined. Finally, it shall be interesting to see possible extensions of the model, both in the direction of scalar conformal modifications that can help to fix the Higgs mass as well as extensions of the Pati-Salam type [32, 33].

A The static spectral action

A.1 Leptonic sector

In the leptonic sector we have

$$\widetilde{D_L}^2 = -(\mathbf{1}_4 \otimes \mathbf{1}_2)\Delta - a^j \partial_j - b, \tag{A.1}$$

where,

$$\begin{aligned}
a^j &= -2i\left(A_j \begin{pmatrix} \sigma^3 - \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_2 & \\ & W_j \end{pmatrix}\right) \otimes \mathbf{1}_2,
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
b = &- \begin{pmatrix} \Phi_l {\Phi_l}^\dagger & \\ & {\Phi_l}^\dagger \Phi_l \end{pmatrix} \otimes \mathbf{1}_2 - A_j A_k \begin{pmatrix} 2(\mathbf{1}_2 - \sigma^3) & \\ & \mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \sigma^k \\
&- \begin{pmatrix} \mathbf{0}_2 & \\ & W_j W_k \end{pmatrix} \otimes \sigma^j \sigma^k + \begin{pmatrix} & \Phi_l W_j \\ W_j {\Phi_l}^\dagger & \end{pmatrix} \otimes \sigma^j + 2 \begin{pmatrix} \mathbf{0}_2 & \\ & W_j \end{pmatrix} A_j \otimes \mathbf{1}_2 \\
&- i \begin{pmatrix} & \partial_j \Phi_l \\ -\partial_j \Phi_l^\dagger & \end{pmatrix} \otimes \sigma^j - i \begin{pmatrix} \mathbf{0}_2 & \\ & \partial_j W_k \end{pmatrix} \otimes \sigma^j \sigma^k \\
&- i \begin{pmatrix} \sigma^3 - \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix} \partial_j A_k \otimes \sigma^j \sigma^k - \begin{pmatrix} & \sigma^3 \Phi_l \\ {\Phi_l}^\dagger \sigma^3 & \end{pmatrix} A_j \otimes \sigma^j .
\end{aligned} \tag{A.3}$$

As a result, following the notation of [20], we have $\omega_j = \frac{1}{2} a^j$, so that

$$\begin{aligned}
\Omega_{ij} &= \partial_i \omega_j - \partial_j \omega_i + \omega_i \omega_j - \omega_j \omega_i \\
&= -iF_{ij} \begin{pmatrix} \sigma^3 - \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix} \otimes \mathbf{1}_2 - i \begin{pmatrix} \mathbf{0}_2 & \\ & W_{ij} \end{pmatrix} \otimes \mathbf{1}_2,
\end{aligned} \tag{A.4}$$

with

$$F_{ij} = \partial_i A_j - \partial_j A_i, \qquad W_{ij} = \partial_i W_j - \partial_j W_i - i[W_i, W_j]. \tag{A.5}$$

Next we compute,

$$\begin{aligned}
E &= b - \partial_j \omega_j - \omega_j \omega_j \\
&= - \begin{pmatrix} \Phi_l \Phi_l^\dagger & \\ & \Phi_l^\dagger \Phi_l \end{pmatrix} \otimes \mathbf{1}_2 - i \begin{pmatrix} & \partial_j \Phi_l \\ -\partial_j \Phi_l^\dagger & \end{pmatrix} \otimes \sigma^j \\
&\quad + \frac{1}{2} \begin{pmatrix} \mathbf{0}_2 & \\ & W_{jk} \end{pmatrix} \otimes \varepsilon^{jkl} \sigma^l + \frac{1}{2} F_{jk} \begin{pmatrix} \sigma^3 - \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix} \otimes \varepsilon^{jkl} \sigma^l \\
&\quad - A_j \begin{pmatrix} & \sigma^3 \Phi_l \\ \Phi_l^\dagger \sigma^3 & \end{pmatrix} \otimes \sigma^j + \begin{pmatrix} & \Phi_l W_j \\ W_j \Phi_l^\dagger & \end{pmatrix} \otimes \sigma^j .
\end{aligned} \tag{A.6}$$

We get then

$$\mathrm{Tr}(E) = -4\kappa \mathrm{Tr}(\Phi_l^\dagger \Phi_l), \tag{A.7}$$

and furthermore,

$$\mathrm{Tr}(\Omega_{ij}\Omega_{ij}) = -2\kappa \left(6F^2 + \mathrm{Tr}(W^2)\right), \tag{A.8}$$

where κ is the normalization of the trace (i.e. everything within the bracket is computed for the unnormalized trace) and $W^2 = W_{jk}W_{jk}$. Moreover,

$$\begin{aligned} \kappa^{-1}\mathrm{Tr}(E^2) = {}& 4\mathrm{Tr}(\Phi_l^\dagger\Phi_l)^2 + 4\mathrm{Tr}[(\partial_j\Phi_l^\dagger)(\partial_j\Phi_l)] + 4A^2\mathrm{Tr}(\Phi_l^\dagger\Phi_l) + 4\mathrm{Tr}(W_jW_j\Phi_l^\dagger\Phi_l) \\ & + 4iA_j\mathrm{Tr}\left[((\partial_j\Phi_l)\Phi_l^\dagger - \Phi_l(\partial_j\Phi_l^\dagger))\sigma^3\right] - 4i\mathrm{Tr}\left[(\Phi_l^\dagger(\partial_j\Phi_l) - (\partial_j\Phi_l^\dagger)\Phi_l)W_j\right] \\ & - 8A_j\mathrm{Tr}\left(\Phi_l^\dagger\sigma^3\Phi_lW_j\right) + 6F^2 + \mathrm{Tr}(W^2). \end{aligned} \tag{A.9}$$

As a result, in the leptonic sector we have

$$\begin{aligned} a_2 &= \frac{1}{(4\pi)^2}\int d^4x\,\mathrm{Tr}E = -\frac{\kappa}{4\pi^2}\int d^4x\,\mathrm{Tr}(\Phi_l^\dagger\Phi_l), \\ a_4 &= \frac{1}{16\pi^2}\frac{1}{12}\int d^4x\,\left(6\mathrm{Tr}(E^2) + \mathrm{Tr}(\Omega_{ij}\Omega_{ij})\right) \\ &= \frac{\kappa}{48\pi^2}\int d^4x\left[6\left(\mathrm{Tr}(\Phi_l^\dagger\Phi_l)^2 + \mathrm{Tr}[(\partial_j\Phi_l^\dagger)(\partial_j\Phi_l)] + A^2\mathrm{Tr}(\Phi_l^\dagger\Phi_l)\right.\right. \\ &\quad +\mathrm{Tr}(W_jW_j\Phi_l^\dagger\Phi_l) + iA_j\mathrm{Tr}\left[((\partial_j\Phi_l)\Phi_l^\dagger - \Phi_l(\partial_j\Phi_l^\dagger))\sigma^3\right] \\ &\quad \left.-i\mathrm{Tr}\left[(\Phi_l^\dagger(\partial_j\Phi_l) - (\partial_j\Phi_l^\dagger)\Phi_l)W_j\right] - 2A_j\mathrm{Tr}\left(\Phi_l^\dagger\sigma^3\Phi_lW_j\right)\right) \\ &\quad \left.+6F^2 + \mathrm{Tr}(W^2)\right]. \end{aligned} \tag{A.10}$$

Using the parametrization from the section 3.1 in this sector we then have

$$\Phi_l^\dagger\Phi_l = \begin{pmatrix} |\Upsilon_\nu|^2|H_1|^2 + |\Upsilon_e|^2|H_2|^2 & |\Upsilon_\nu|^2\overline{H_1}H_2 - |\Upsilon_e|^2H_2\overline{H_1} \\ |\Upsilon_\nu|^2\overline{H_2}H_1 - |\Upsilon_e|^2H_1\overline{H_2} & |\Upsilon_\nu|^2|H_2|^2 + |\Upsilon_e|^2|H_1|^2 \end{pmatrix},$$

and as a result

$$a_2 = -\frac{\kappa}{4\pi^2}(|\Upsilon_e|^2 + |\Upsilon_\nu|^2)\int d^4x\,|H|^2.$$

Furthermore we have

$$\begin{aligned} \mathrm{Tr}(\Phi_l^\dagger\Phi_l)^2 &= (|\Upsilon_\nu|^4 + |\Upsilon_e|^4)|H|^4, \\ \mathrm{Tr}\left[(\partial_j\Phi_l^\dagger)(\partial_j\Phi_l)\right] &= (|\Upsilon_\nu|^2 + |\Upsilon_e|^2)\,|\partial_jH|^2, \\ (A_jA_j)\mathrm{Tr}(\Phi_l^\dagger\Phi_l) &= (A_jA_j)(|\Upsilon_\nu|^2 + |\Upsilon_e|^2)\,|H|^2. \end{aligned}$$

Next, we decompose the W field in terms of Pauli matrices,

$$W_j = W_{j,1}\sigma^1 + W_{j,2}\sigma^2 + W_{j,3}\sigma^3,$$

so that

$$\mathrm{Tr}(W_jW_j\Phi_l^\dagger\Phi_l) = (W_jW_j)(|\Upsilon_\nu|^2 + |\Upsilon_e|^2)|H|^2.$$

Next, we compute

$$
\begin{aligned}
iA_j\mathrm{Tr}\left[((\partial_j\Phi_l)\Phi_l^\dagger-\Phi_l(\partial_j\Phi_l^\dagger))\sigma^3\right]&=iA_j(|\Upsilon_\nu|^2+|\Upsilon_e|^2)(H^\dagger\partial_jH-\partial_jH^\dagger H),\\
-i\mathrm{Tr}\left[(\Phi_l^\dagger(\partial_j\Phi_l)-(\partial_j\Phi_l^\dagger)\Phi_l)W_j\right]&=-i(|\Upsilon_\nu|^2+|\Upsilon_e|^2)\left[W_{j,3}\left(\overline{H_1}\partial_jH_1-H_1\partial_j\overline{H_1}-\overline{H_2}\partial_jH_2+H_2\partial_j\overline{H_2}\right)\right.\\
&\quad+(W_{j,1}-iW_{j,2})(\overline{H_2}\partial_jH_1-H_1\partial_j\overline{H_2})\\
&\quad\left.+(W_{j,1}+iW_{j,2})(\overline{H_1}\partial_jH_2-H_2\partial_j\overline{H_1})\right],\\
-2A_j\mathrm{Tr}\left[\Phi_l^\dagger\sigma^3\Phi_lW_j\right]&=-2A_j(|\Upsilon_\nu|^2+|\Upsilon_e|^2)\left[(W_{j,1}-iW_{j,2})H_1\overline{H_2}\right.\\
&\quad\left.+(W_{j,1}+iW_{j,2})\overline{H_1}H_2+W_{j,3}(|H_1|^2-|H_2|^2)\right].
\end{aligned}
$$

Let us now verify whether these terms can be written using the covariant derivative of the Higgs doublet,

$$D_jH=\partial_jH+iW_jH-iA_jH.$$

We check,

$$
\begin{aligned}
\mathrm{Tr}|D_jH|^2=\mathrm{Tr}\big[&|\partial_jH|^2+i(\partial_jH^\dagger W_jH-H^\dagger W_j\partial_jH)\\
&+iA_j(H^\dagger\partial_jH-\partial_jH^\dagger H)-2A_jH^\dagger W_jH+W_jW_j|H|^2+A^2|H|^2\big].
\end{aligned}
$$

The only terms that are potentially different that the ones in the coefficient a_4 are

$$2A_j\mathrm{Tr}(H^\dagger W_jH),\qquad i\mathrm{Tr}(\partial_jH^\dagger W_jH-H^\dagger W_j\partial_jH),$$

but simple calculation shows that

$$
\begin{aligned}
A_j\mathrm{Tr}(H^\dagger W_jH)=A_j\big[&W_{j,1}(\overline{H_1}H_2-\overline{H_2}H_1)\\
&+iW_{j,2}(\overline{H_1}H_2-\overline{H_2}H_1)+W_{j,3}(|H_1|^2-|H_2|^2)\big],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Tr}(\partial_jH^\dagger W_jH-H^\dagger W_j\partial_jH)=&\,W_{j,1}(\partial_j\overline{H_1}H_2+\partial_j\overline{H_2}H_1-\overline{H_1}\partial_jH_2-\overline{H_2}\partial_jH_1)\\
&+W_{j,2}(\partial_j\overline{H_1}H_2-\partial_j\overline{H_2}H_1-\overline{H_1}\partial_jH_2+\overline{H_2}\partial_jH_1)\\
&+W_{j,3}(\partial_j\overline{H_1}H_1-\partial_j\overline{H_2}H_2-\overline{H_1}\partial_jH_1+\overline{H_2}\partial_jH_2)
\end{aligned}
$$

in a complete agreement with a_4.

Therefore,

$$a_4=\frac{\kappa}{8\pi^2}\int d^4x\,\left[(|\Upsilon_\nu|^4+|\Upsilon_e|^4)|H|^4+(|\Upsilon_\nu|^2+|\Upsilon_e|^2)\mathrm{Tr}|D_jH|^2+F^2+\frac{1}{6}\mathrm{Tr}W^2\right].\qquad\text{(A.11)}$$

A.2 Quark sector

In this sector we have

$$\widetilde{D}_Q^2=-(\mathbf{1}_4\otimes\mathbf{1}_2\otimes\mathbf{1}_3)\Delta-a^j\partial_j-b,\qquad\text{(A.12)}$$

where

$$a^j = -2i\left[A_j\begin{pmatrix}\sigma^3+\frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3+\begin{pmatrix}\mathbf{0}_2 & \\ & W_j\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3+\mathbf{1}_4\otimes\mathbf{1}_2\otimes G_j\right],$$

$$\begin{aligned}
b = &-\begin{pmatrix}\Phi_q\Phi_q^\dagger & \\ & \Phi_q^\dagger\Phi_q\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 - A_jA_k\begin{pmatrix}\frac{8}{9}\mathbf{1}_2 & \\ & -\frac{1}{9}\mathbf{1}_2\end{pmatrix}\otimes\sigma^j\sigma^k\otimes\mathbf{1}_3\\
&-\begin{pmatrix}\mathbf{0}_2 & \\ & W_jW_k\end{pmatrix}\otimes\sigma^j\sigma^k\otimes\mathbf{1}_3-\mathbf{1}_4\otimes\sigma^j\sigma^k\otimes G_jG_k\\
&-i\begin{pmatrix} & \partial_j\Phi_q\\ -\partial_j\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3-i\begin{pmatrix}\mathbf{0}_2 & \\ & \partial_jW_k\end{pmatrix}\otimes\sigma^j\sigma^k\otimes\mathbf{1}_3\\
&-i1_4\otimes\sigma^j\sigma^k\otimes\partial_jG_k-A_j\begin{pmatrix} & \sigma^3\Phi_q\\ \Phi_q^\dagger\sigma^3 & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3\\
&+\begin{pmatrix} & \Phi_qW_j\\ W_j\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3-i(\partial_jA_k)\begin{pmatrix}\sigma^3+\frac{1}{2}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2\end{pmatrix}\otimes\sigma^j\sigma^k\otimes\mathbf{1}_3\\
&-2A_j\begin{pmatrix}\sigma^3+\frac{1}{2}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2\otimes G_j-2\begin{pmatrix}\mathbf{0}_2 & \\ & W_j\end{pmatrix}\otimes\mathbf{1}_2\otimes G_j\\
&-\frac{2}{3}A_jA_k\begin{pmatrix}\sigma^3 & \\ & \mathbf{0}_2\end{pmatrix}\otimes\sigma^j\sigma^k\otimes\mathbf{1}_3-\frac{2}{3}\begin{pmatrix}\mathbf{0}_2 & \\ & A_jW_j\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E &= b-\partial_j\omega_j-\omega_j\omega_j\\
&=-\begin{pmatrix}\Phi_q\Phi_q^\dagger & \\ & \Phi_q^\dagger\Phi_q\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3-i\begin{pmatrix} & \partial_j\Phi_q\\ -\partial_j\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3\\
&\quad+\frac{1}{2}F_{jk}\begin{pmatrix}\sigma^3 & \\ & \mathbf{0}_2\end{pmatrix}\otimes\varepsilon^{jkl}\sigma^l\otimes\mathbf{1}_3+\frac{1}{2}\begin{pmatrix}\mathbf{0}_2 & \\ & W_{jk}\end{pmatrix}\otimes\varepsilon^{jkl}\sigma^l\otimes\mathbf{1}_3+\frac{1}{2}\mathbf{1}_4\otimes\varepsilon^{jkl}\sigma^l\otimes G_{jk}\\
&\quad-A_j\begin{pmatrix} & \sigma^3\Phi_q\\ \Phi_q^\dagger\sigma^3 & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3+\begin{pmatrix} & \Phi_qW_j\\ W_j\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3\\
&\quad+\frac{1}{6}F_{jk}\mathbf{1}_4\otimes\varepsilon^{jkl}\sigma^l\otimes\mathbf{1}_3,
\end{aligned}\tag{A.13}$$

where again

$$G_{ij}=\partial_iG_j-\partial_jG_i-i[G_i,G_j].\tag{A.14}$$

As a result,

$$\kappa^{-1}\mathrm{Tr}(E)=-12\mathrm{Tr}(\Phi_q^\dagger\Phi_q).\tag{A.15}$$

and

$$\kappa^{-1}\mathrm{Tr}(\Omega_{ij}\Omega_{ij})=-2\left(\frac{22}{3}F^2+3\mathrm{Tr}(W^2)+4\mathrm{Tr}(G^2)\right),\tag{A.16}$$

where we use short notation $G^2 = G_{ij}G_{ij}$, and the full second contribution reads,

$$\begin{aligned}\kappa^{-1}\mathrm{Tr}(E^2) = & 12\mathrm{Tr}(\Phi_q^\dagger\Phi_q)^2 + 12\mathrm{Tr}[(\partial_j\Phi_q^\dagger)(\partial_j\Phi_q)] + 12A^2\mathrm{Tr}(\Phi_q^\dagger\Phi_q) \\ & + 12\mathrm{Tr}(W_jW_j\Phi_q^\dagger\Phi_q) + 12iA_j\mathrm{Tr}\left[((\partial_j\Phi_q)\Phi_q^\dagger - \Phi_q(\partial_j\Phi_q^\dagger))\sigma^3\right] \\ & - 12i\mathrm{Tr}\left[(\Phi_q^\dagger(\partial_j\Phi_q) - (\partial_j\Phi_q^\dagger)\Phi_q)W_j\right] - 24A_j\mathrm{Tr}\left(\Phi_q^\dagger\sigma^3\Phi_qW_j\right) \\ & + \frac{22}{3}F^2 + 3\mathrm{Tr}(W^2) + 4\mathrm{Tr}(G^2).\end{aligned} \tag{A.17}$$

As a result, in the quark sector we have

$$a_2 = -\frac{\kappa}{4\pi^2}\int d^4x\, 3\mathrm{Tr}(\Phi_q^\dagger\Phi_q), \tag{A.18}$$

$$\begin{aligned}a_4 = \frac{\kappa}{48\pi^2}\int d^4x & \Big[18\Big(\mathrm{Tr}(\Phi_q^\dagger\Phi_q)^2 + \mathrm{Tr}[(\partial_j\Phi_q^\dagger)(\partial_j\Phi_q)] + A^2\mathrm{Tr}(\Phi_q^\dagger\Phi_q) \\ & + \mathrm{Tr}(W_jW_j\Phi_q^\dagger\Phi_q) + iA_j\mathrm{Tr}\left[((\partial_j\Phi_q)\Phi_q^\dagger - \Phi_q(\partial_j\Phi_q^\dagger))\sigma^3\right] \\ & - i\mathrm{Tr}\left[(\Phi_q^\dagger(\partial_j\Phi_q) - (\partial_j\Phi_q^\dagger)\Phi_q)W_j\right] - 2A_j\mathrm{Tr}\left(\Phi_q^\dagger\sigma^3\Phi_qW_j\right)\Big). \\ & + \frac{22}{3}F^2 + 3\mathrm{Tr}(W^2) + 4\mathrm{Tr}(G^2)\Big].\end{aligned} \tag{A.19}$$

In a completely similar manner as for the leptonic sector we derive:

$$a_2 = -\frac{\kappa}{4\pi^2}(3|\Upsilon_u|^2 + 3|\Upsilon_d|^2)\int d^4x|H|^2 \tag{A.20}$$

and

$$\begin{aligned}a_4 = \frac{\kappa}{8\pi^2}\int d^4x & \left[\left(3|\Upsilon_u|^4 + 3|\Upsilon_d|^4\right)|H|^4 + (3|\Upsilon_u|^2 + 3|\Upsilon_d|^2)\mathrm{Tr}|D_jH|^2 \right. \\ & \left. + \frac{11}{9}F^2 + \frac{1}{2}\mathrm{Tr}W^2 + \frac{2}{3}\mathrm{Tr}G^2\right].\end{aligned} \tag{A.21}$$

B The Wick rotated model

B.1 Leptonic sector

Starting with the Wick rotated Dirac operator (3.4) we get

$$\begin{aligned}D_{L,w}^\dagger D_{L,w} = & -(\mathbf{1}_4\otimes\mathbf{1}_2)\Delta_{\mathrm{E}} + 2i\left[A_0\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & -\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2 + \begin{pmatrix}\mathbf{0}_2 & \\ & W_0\end{pmatrix}\otimes\mathbf{1}_2 + \begin{pmatrix} & \Phi_l \\ \Phi_l^\dagger & \end{pmatrix}\otimes\mathbf{1}_2\right]\partial_0 \\ & + 2i\left[A_j\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & -\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2 + \begin{pmatrix}\mathbf{0}_2 & \\ & W_j\end{pmatrix}\otimes\mathbf{1}_2 + \begin{pmatrix} & -i\Phi_l \\ i\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^j\right]\partial_j \\ & + i(\partial_0A_0)\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & -\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2 + i(\partial_jA_k)\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & -\mathbf{1}_2\end{pmatrix}\otimes\sigma^j\sigma^k \\ & + A_0^2\begin{pmatrix}2(\mathbf{1}_2-\sigma^3) & \\ & \mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2 + A_jA_k\begin{pmatrix}2(\mathbf{1}_2-\sigma^3) & \\ & \mathbf{1}_2\end{pmatrix}\otimes\sigma^j\sigma^k\end{aligned}$$

$$
\begin{aligned}
&+\begin{pmatrix}\mathbf{0}_2 & \\ & W_0^2+i\partial_0 W_0\end{pmatrix}\otimes\mathbf{1}_2+\begin{pmatrix}\mathbf{0}_2 & \\ & W_jW_k+i\partial_jW_k\end{pmatrix}\otimes\sigma^j\sigma^k\\
&-F_{0j}\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & \mathbf{1}_2\end{pmatrix}\otimes\sigma^j+\begin{pmatrix}\mathbf{0}_2 & \\ & W_{0j}\end{pmatrix}\otimes\sigma^j-2\begin{pmatrix}\mathbf{0}_2 & \\ & A_0W_0+A_jW_j\end{pmatrix}\otimes\mathbf{1}_2\\
&+i\begin{pmatrix} & \partial_0\Phi_l\\ \partial_0\Phi_l^\dagger & \end{pmatrix}\otimes\mathbf{1}_2+i\begin{pmatrix} & -i\partial_j\Phi_l\\ i\partial_j\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^j+\begin{pmatrix}\Phi_l\Phi_l^\dagger & \\ & \Phi_l^\dagger\Phi_l\end{pmatrix}\otimes\mathbf{1}_2\\
&+A_0\begin{pmatrix} & (\sigma^3-2\cdot\mathbf{1}_2)\Phi_l\\ \Phi_l^\dagger(\sigma^3-2\cdot\mathbf{1}_2) & \end{pmatrix}\otimes\mathbf{1}_2+\begin{pmatrix} & \Phi_lW_0\\ W_0\Phi_l^\dagger & \end{pmatrix}\otimes\mathbf{1}_2\\
&+A_j\begin{pmatrix} & -i(\sigma^3-2\cdot\mathbf{1}_2)\Phi_l\\ i\Phi_l^\dagger(\sigma^3-2\cdot\mathbf{1}_2) & \end{pmatrix}\otimes\sigma^j+\begin{pmatrix} & -i\Phi_lW_j\\ iW_j\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^j.
\end{aligned}
$$

Writing $D_{L,w}^\dagger D_{L,w}$ in the canonical form $-(\mathbf{1}_4\otimes\mathbf{1}_2)\Delta_{\mathrm{E}}-2\omega_\mu\partial_\mu-b$ (with the Euclidean summation) we get

$$
\begin{aligned}
E=&\frac{1}{2}F_{jk}\varepsilon^{jkl}\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & -\mathbf{1}_2\end{pmatrix}\otimes\sigma^l+F_{0j}\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & \mathbf{1}_2\end{pmatrix}\otimes\sigma^j\\
&+\frac{1}{2}\varepsilon^{jkl}\begin{pmatrix}\mathbf{0}_4 & \\ & W_{jk}\end{pmatrix}\otimes\sigma^l-\begin{pmatrix}\mathbf{0}_4 & \\ & W_{0j}\end{pmatrix}\otimes\sigma^j+3\begin{pmatrix}\Phi_l\Phi_l^\dagger & \\ & \Phi_l^\dagger\Phi_l\end{pmatrix}\otimes\mathbf{1}_2.
\end{aligned}
\tag{B.1}
$$

Its trace is therefore

$$
\mathrm{Tr}(E)=12\kappa\mathrm{Tr}(\Phi_l^\dagger\Phi_l^\dagger).
\tag{B.2}
$$

Furthermore, we have

$$
\kappa^{-1}\mathrm{Tr}(E^2)=6F^2+\mathrm{Tr}(W^2)+36\mathrm{Tr}(\Phi_l^\dagger\Phi_l)^2+4\varepsilon^{jkl}F_{jk}F_{0l}-2\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l}),
\tag{B.3}
$$

where now $F^2=F_{\mu\nu}F_{\mu\nu}=F_{jk}F_{jk}+2F_{0j}F_{0j}$ and similarly for W^2.

Next, we have

$$
\begin{aligned}
\Omega_{0j}=&-iF_{0j}\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & -\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2-i\begin{pmatrix}\mathbf{0}_2 & \\ & W_{0j}\end{pmatrix}\otimes\mathbf{1}_2+iA_0\begin{pmatrix} & \sigma^3\Phi_l\\ \Phi_l^\dagger\sigma^3 & \end{pmatrix}\otimes\sigma^j\\
&-i\begin{pmatrix} & \Phi_lW_0\\ W_0\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^j-A_j\begin{pmatrix} & -\sigma^3\Phi_l\\ \Phi_l^\dagger\sigma^3 & \end{pmatrix}\otimes\mathbf{1}_2+\begin{pmatrix} & -\Phi_lW_j\\ W_j\Phi_l^\dagger & \end{pmatrix}\otimes\mathbf{1}_2\\
&-2i\begin{pmatrix}\Phi_l\Phi_l^\dagger & \\ & -\Phi_l^\dagger\Phi_l\end{pmatrix}\otimes\sigma^j+i\begin{pmatrix} & \partial_j\Phi_l\\ \partial_j\Phi_l^\dagger & \end{pmatrix}\otimes\mathbf{1}_2+\begin{pmatrix} & -\partial_0\Phi_l\\ \partial_0\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^j,
\end{aligned}
\tag{B.4}
$$

hence

$$
\begin{aligned}
\kappa^{-1}\mathrm{Tr}(\Omega_{0j}\Omega_{0j})=&-12F_{0j}F_{0j}-2\mathrm{Tr}(W_{0j}W_{0j})-12A_0^2\mathrm{Tr}(\Phi_l^\dagger\Phi_l)-12\mathrm{Tr}(W_0^2\Phi_l^\dagger\Phi_l)\\
&-4A_j^2\mathrm{Tr}(\Phi_l^\dagger\Phi_l)-4\mathrm{Tr}\left(W_j^2\Phi_l^\dagger\Phi_l\right)-48\mathrm{Tr}(\Phi_l^\dagger\Phi_l)^2+24A_0\mathrm{Tr}(\Phi_l^\dagger\sigma^3\Phi_lW_0)\\
&+8A_j\mathrm{Tr}(\Phi_l^\dagger\sigma^3\Phi_lW_j)-4\mathrm{Tr}\left[(\partial_j\Phi_l)^\dagger(\partial_j\Phi_l)\right]-12\mathrm{Tr}\left[(\partial_0\Phi_l)^\dagger(\partial_0\Phi_l)\right]\\
&-12iA_0\mathrm{Tr}\left[\left((\partial_0\Phi_l)\Phi_l^\dagger-\Phi_l(\partial_0\Phi_l^\dagger)\right)\sigma^3\right]-4iA_j\mathrm{Tr}\left[\left((\partial_j\Phi_l)\Phi_l^\dagger-\Phi_l(\partial_j\Phi_l^\dagger)\right)\sigma^3\right]\\
&+12i\mathrm{Tr}\left[\left(\Phi_l^\dagger(\partial_0\Phi_l)-(\partial_0\Phi_l^\dagger)\Phi_l\right)W_0\right]+4i\mathrm{Tr}\left[\left(\Phi_l^\dagger(\partial_j\Phi_l)-(\partial_j\Phi_l^\dagger)\Phi_l\right)W_j\right].
\end{aligned}
\tag{B.5}
$$

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Moreover,

$$
\begin{aligned}
\Omega_{jk} = &-iF_{jk}\begin{pmatrix}\sigma^3-\mathbf{1}_2 & \\ & -\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2 - i\begin{pmatrix}\mathbf{0}_2 & \\ & W_{jk}\end{pmatrix}\otimes\mathbf{1}_2 + \begin{pmatrix} & -\partial_j\Phi_l \\ \partial_j\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^k \\
&-\begin{pmatrix} & -\partial_k\Phi_l \\ \partial_k\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^j - 2i\varepsilon^{jkl}\begin{pmatrix}\Phi_l\Phi_l^\dagger & \\ & \Phi_l^\dagger\Phi_l\end{pmatrix}\otimes\sigma^l - i\begin{pmatrix} & \Phi_l W_j \\ W_j\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^k \\
&+i\begin{pmatrix} & \Phi_l W_k \\ W_k\Phi_l^\dagger & \end{pmatrix}\otimes\sigma^j + i\begin{pmatrix} & \sigma^3\Phi_l \\ \Phi_l^\dagger\sigma^3 & \end{pmatrix}\otimes(A_j\sigma^k - A_k\sigma^j).
\end{aligned}
\tag{B.6}
$$

so that

$$
\begin{aligned}
\kappa^{-1}\mathrm{Tr}(\Omega_{jk}\Omega_{jk}) = &-12F_{jk}F_{jk} - 2\mathrm{Tr}(W_{jk}W_{jk}) - 96\mathrm{Tr}(\Phi_l^\dagger\Phi_l)^2 \\
&-16\mathrm{Tr}\left[(\partial_j\Phi_l^\dagger)(\partial_j\Phi_l)\right] - 16\mathrm{Tr}(W_j^2\Phi_l^\dagger\Phi_l) - 16A_j^2\mathrm{Tr}(\Phi_l^\dagger\Phi_l) \\
&+32A_j\mathrm{Tr}(\Phi_l W_j\Phi_l^\dagger\sigma^3) - 16i\mathrm{Tr}\left[\left((\partial_j\Phi_l^\dagger)\Phi_l - \Phi_l^\dagger(\partial_j\Phi_l)\right)W_j\right] \\
&+16iA_j\mathrm{Tr}\left[\left(\Phi_l(\partial_j\Phi_l^\dagger) - (\partial_j\Phi_l)\Phi_l^\dagger\right)\sigma^3\right].
\end{aligned}
\tag{B.7}
$$

Therefore,

$$
\begin{aligned}
\kappa^{-1}\mathrm{Tr}(\Omega^2) = &-12F^2 - 2\mathrm{Tr}(W^2) - 24A_\mu A_\mu\mathrm{Tr}(\Phi_l^\dagger\Phi_l) - 24\mathrm{Tr}(W_\mu W_\mu\Phi_l^\dagger\Phi_l) \\
&-192\mathrm{Tr}(\Phi_l^\dagger\Phi_l)^2 + 48A_\mu\mathrm{Tr}\left(\Phi_l^\dagger\sigma^3\Phi_l W_\mu\right) - 24\mathrm{Tr}\left[(\partial_\mu\Phi_l)^\dagger(\partial_\mu\Phi_l)\right] \\
&-24iA_\mu\mathrm{Tr}\left[\left((\partial_\mu\Phi_l)\Phi_l^\dagger - \Phi_l(\partial_\mu\Phi_l^\dagger)\right)\sigma^3\right] \\
&+24i\mathrm{Tr}\left[\left(\Phi_l^\dagger(\partial_\mu\Phi_l) - (\partial_\mu\Phi_l^\dagger)\Phi_l\right)W_\mu\right],
\end{aligned}
\tag{B.8}
$$

where the summation is performed over Euclidean indices.

As a result, in the leptonic sector we have

$$
\begin{aligned}
a_2 &= \frac{3\kappa}{4\pi^2}\int d^4x\,\mathrm{Tr}(\Phi_l^\dagger\Phi_l), \\
a_4 &= \frac{\kappa}{48\pi^2}\int d^4x\left[6\left(\mathrm{Tr}(\Phi_l^\dagger\Phi_l)^2 - \mathrm{Tr}[(\partial_\mu\Phi_l^\dagger)(\partial_\mu\Phi_l)] - A^2\mathrm{Tr}(\Phi_l^\dagger\Phi_l)\right.\right. \\
&\quad -\mathrm{Tr}(W_\mu W_\mu\Phi_l^\dagger\Phi_l) - iA_\mu\mathrm{Tr}\left[((\partial_\mu\Phi_l)\Phi_l^\dagger - \Phi_l(\partial_\mu\Phi_l^\dagger))\sigma^3\right] \\
&\quad \left. +i\mathrm{Tr}\left[(\Phi_l^\dagger(\partial_\mu\Phi_l) - (\partial_\mu\Phi_l^\dagger)\Phi_l)W_\mu\right] + 2A_\mu\mathrm{Tr}\left(\Phi_l^\dagger\sigma^3\Phi_l W_\mu\right)\right) \\
&\quad \left. +6F^2 + \mathrm{Tr}(W^2) + 6\varepsilon^{jkl}F_{jk}F_{0l} - 3\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l})\right].
\end{aligned}
\tag{B.9}
$$

Using the parametrization from section 3.1 we can further write

$$
a_2 = \frac{3\kappa}{4\pi^2}(|\Upsilon_e|^2 + |\Upsilon_\nu|^2)\int d^4x|H|^2, \tag{B.10}
$$

and

$$
\begin{aligned}
a_4 = \frac{\kappa}{8\pi^2}\int d^4x&\left[(|\Upsilon_\nu|^4 + |\Upsilon_e|^4)|H|^4 - (|\Upsilon_\nu|^2 + |\Upsilon_e|^2)\mathrm{Tr}|D_\mu H|^2\right. \\
&\left. +F^2 + \frac{1}{6}\mathrm{Tr}(W^2) + \varepsilon^{jkl}F_{jk}F_{0l} - \frac{1}{2}\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l})\right].
\end{aligned}
\tag{B.11}
$$

B.2 Quark sector

For the quark sector, starting from (3.7), we get

$$
\begin{aligned}
D^\dagger_{Q,w} D_{Q,w} = -\Delta_{\mathrm{E}} + 2i \Bigg[& A_0 \begin{pmatrix} \sigma^3 + \frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2 \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 + \begin{pmatrix} \mathbf{0}_2 & \\ & W_0 \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 \\
& + \mathbf{1}_4 \otimes \mathbf{1}_2 \otimes G_0 + \begin{pmatrix} & \Phi_q \\ \Phi_q^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 \Bigg] \partial_0 + 2i \Bigg[A_j \begin{pmatrix} \sigma^3 + \frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2 \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 \\
& + \begin{pmatrix} \mathbf{0}_2 & \\ & W_j \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 + \mathbf{1}_4 \otimes \mathbf{1}_2 \otimes G_j + \begin{pmatrix} & -i\Phi_q \\ i\Phi_q^\dagger & \end{pmatrix} \otimes \sigma^j \otimes \mathbf{1}_3 \Bigg] \partial_j \\
& + i(\partial_0 A_0) \begin{pmatrix} \sigma^3 + \frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2 \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 + i(\partial_j A_k) \begin{pmatrix} \sigma^3 + \frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \sigma^k \otimes \mathbf{1}_3 \\
& + A_0^2 \begin{pmatrix} \frac{2}{3}\sigma^3 + \frac{10}{9}\mathbf{1}_2 & \\ & \frac{1}{9}\mathbf{1}_2 \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 + A_j A_k \begin{pmatrix} \frac{2}{3}\sigma^3 + \frac{10}{9}\mathbf{1}_2 & \\ & \frac{1}{9}\mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \sigma^k \otimes \mathbf{1}_3 \\
& + \begin{pmatrix} \mathbf{0}_2 & \\ & W_0^2 + i\partial_0 W_0 \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 + \begin{pmatrix} \mathbf{0}_2 & \\ & W_j W_k + i\partial_j W_k \end{pmatrix} \otimes \sigma^j \sigma^k \otimes \mathbf{1}_3 \\
& + \mathbf{1}_4 \otimes \mathbf{1}_2 \otimes (G_0^2 + i\partial_0 G_0) + \mathbf{1}_4 \otimes \sigma^j \sigma^k \otimes (G_j G_k + i\partial_j G_k) \\
& - F_{0j} \begin{pmatrix} \sigma^3 + \frac{1}{3}\mathbf{1}_2 & \\ & -\frac{1}{3}\mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \otimes \mathbf{1}_3 + \begin{pmatrix} \mathbf{0}_2 & \\ & W_{0j} \end{pmatrix} \otimes \sigma^j \otimes \mathbf{1}_3 - \begin{pmatrix} \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix} \otimes \sigma^j \otimes G_{0j} \\
& + \frac{2}{3} \begin{pmatrix} \mathbf{0}_2 & \\ & A_0 W_0 + A_j W_j \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 + 2 \begin{pmatrix} \sigma^3 + \frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2 \end{pmatrix} \otimes \mathbf{1}_2 \otimes (A_0 G_0 + A_j G_j) \\
& + 2 \begin{pmatrix} \mathbf{0}_2 & \\ & W_0 \end{pmatrix} \otimes \mathbf{1}_2 \otimes G_0 + 2 \begin{pmatrix} \mathbf{0}_2 & \\ & W_j \end{pmatrix} \otimes \mathbf{1}_2 \otimes G_j + i \begin{pmatrix} & \partial_0 \Phi_q \\ \partial_0 \Phi_q^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 \\
& + \begin{pmatrix} & \partial_j \Phi_q \\ -\partial_j \Phi_q^\dagger & \end{pmatrix} \otimes \sigma^j \otimes \mathbf{1}_3 + A_0 \begin{pmatrix} & \sigma^3 \Phi_q + \frac{2}{3}\Phi_q \\ \Phi_q^\dagger \sigma^3 + \frac{2}{3}\Phi_q^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 \\
& - iA_j \begin{pmatrix} & \sigma^3 \Phi_q + \frac{2}{3}\Phi_q \\ -\Phi_q^\dagger \sigma^3 - \frac{2}{3}\Phi_q^\dagger & \end{pmatrix} \otimes \sigma^j \otimes \mathbf{1}_3 + \begin{pmatrix} & \Phi_q W_0 \\ W_0 \Phi_q^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 \\
& + i \begin{pmatrix} & -\Phi_q W_j \\ W_j \Phi_q^\dagger & \end{pmatrix} \otimes \sigma^j \otimes \mathbf{1}_3 + 2 \begin{pmatrix} & \Phi_q \\ \Phi_q^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 \otimes G_0 \\
& + 2i \begin{pmatrix} & -\Phi_q \\ \Phi_q^\dagger & \end{pmatrix} \otimes \sigma^j \otimes G_j + \begin{pmatrix} \Phi_q \Phi_q^\dagger & \\ & \Phi_q^\dagger \Phi_q \end{pmatrix} \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 .
\end{aligned}
\tag{B.12}
$$

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In this case we therefore have

$$E = \frac{1}{2}F_{jk}\varepsilon^{jkl}\begin{pmatrix}\sigma^3+\frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2\end{pmatrix}\otimes\sigma^l\otimes\mathbf{1}_3 + F_{0j}\begin{pmatrix}\sigma^3+\frac{1}{3}\mathbf{1}_2 & \\ & -\frac{1}{3}\mathbf{1}_2\end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3$$
$$+\frac{1}{2}\varepsilon^{jkl}\begin{pmatrix}\mathbf{0}_2 & \\ & W_{jk}\end{pmatrix}\otimes\sigma^l\otimes\mathbf{1}_3 - \begin{pmatrix}\mathbf{0}_2 & \\ & W_{0j}\end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3$$
$$+\frac{1}{2}\varepsilon^{jkl}\mathbf{1}_4\otimes\sigma^l\otimes G_{jk} + \begin{pmatrix}\mathbf{1}_2 & \\ & -\mathbf{1}_2\end{pmatrix}\otimes\sigma^j\otimes G_{0j}$$
$$+3\begin{pmatrix}\Phi_q\Phi_q^\dagger & \\ & \Phi_q^\dagger\Phi_q\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3. \tag{B.13}$$

Hence

$$\mathrm{Tr}(E) = 36\kappa\,\mathrm{Tr}(\Phi_q^\dagger\Phi_q). \tag{B.14}$$

Furthermore, we have

$$\mathrm{Tr}(E^2) = \frac{22}{3}F^2 + 3\mathrm{Tr}(W^2) + 4\mathrm{Tr}(G^2) + 12\varepsilon^{jkl}F_{jk}F_{0l} - 6\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l}) + 108\mathrm{Tr}(\Phi_q^\dagger\Phi_q)^2. \tag{B.15}$$

Moreover,

$$\Omega_{0j} = -F_{0j}\begin{pmatrix}\sigma^3+\frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 - i\begin{pmatrix}\mathbf{0}_2 & \\ & W_{0j}\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 - i\mathbf{1}_4\otimes\mathbf{1}_2\otimes G_{0j}$$
$$+iA_0\begin{pmatrix} & \sigma^3\Phi_q \\ \Phi_q^\dagger\sigma^3 & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3 - i\begin{pmatrix} & \Phi_q W_0 \\ W_0\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3$$
$$-A_j\begin{pmatrix} & -\sigma^3\Phi_q \\ \Phi_q^\dagger\sigma^3 & \end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 + \begin{pmatrix} & -\Phi_q W_j \\ W_j\Phi_q^\dagger & \end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3$$
$$-2i\begin{pmatrix}\Phi_q\Phi_q^\dagger & \\ & -\Phi_q^\dagger\Phi_q\end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3 + i\begin{pmatrix} & \partial_j\Phi_q \\ \partial_j\Phi_q^\dagger & \end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3$$
$$+\begin{pmatrix} & -\partial_0\Phi_q \\ \partial_0\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3, \tag{B.16}$$

hence

$$\kappa^{-1}\mathrm{Tr}(\Omega_{0j}\Omega_{0j}) = -\frac{44}{3}F_{0j}F_{0j} - 6\mathrm{Tr}(W_{0j}W_{0j}) - 8\mathrm{Tr}(G_{0j}G_{0j}) - 36A_0^2\mathrm{Tr}(\Phi_q^\dagger\Phi_q)$$
$$-36\mathrm{Tr}(W_0^2\Phi_q^\dagger\Phi_q) - 12A_j^2\mathrm{Tr}(\Phi_q^\dagger\Phi_q) - 12\mathrm{Tr}(W_j^2\Phi_q^\dagger\Phi_q) - 144\mathrm{Tr}(\Phi_q^\dagger\Phi_q)^2$$
$$+72A_0\mathrm{Tr}(\Phi_q^\dagger\sigma^3\Phi_q W_0) + 24A_j\mathrm{Tr}(\Phi_q^\dagger\sigma^3\Phi_q W_j) - 12\mathrm{Tr}\left[(\partial_j\Phi_q)^\dagger(\partial_j\Phi_q)\right]$$
$$-36\mathrm{Tr}\left[(\partial_0\Phi_q)^\dagger(\partial_0\Phi_q)\right] - 36iA_0\mathrm{Tr}\left[\left((\partial_0\Phi_q)\Phi_q^\dagger - \Phi_q(\partial_0\Phi_q)^\dagger\right)\sigma^3\right]$$
$$-12iA_j\mathrm{Tr}\left[\left((\partial_j\Phi_q)\Phi_q^\dagger - \Phi_q(\partial_j\Phi_q)^\dagger\right)\sigma^3\right] + 36i\mathrm{Tr}\left[\left(\Phi_q^\dagger(\partial_0\Phi_q) - (\partial_0\Phi_q^\dagger)\Phi_q\right)W_0\right]$$
$$+12i\mathrm{Tr}\left[\left(\Phi_q^\dagger(\partial_j\Phi_q) - (\partial_j\Phi_q)^\dagger\Phi_q\right)W_j\right]. \tag{B.17}$$

Moreover,

$$\Omega_{jk} = -iF_{jk}\begin{pmatrix}\sigma^3+\frac{1}{3}\mathbf{1}_2 & \\ & \frac{1}{3}\mathbf{1}_2\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 - i\begin{pmatrix}\mathbf{0}_2 & \\ & W_{jk}\end{pmatrix}\otimes\mathbf{1}_2\otimes\mathbf{1}_3 - i\mathbf{1}_4\otimes\mathbf{1}_2\otimes G_{jk}+$$
$$+\begin{pmatrix} & -\partial_j\Phi_q \\ \partial_j\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^k\otimes\mathbf{1}_3 - \begin{pmatrix} & -\partial_k\Phi_q \\ \partial_k\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3$$
$$-2i\varepsilon^{jkl}\begin{pmatrix}\Phi_q\Phi_q^\dagger & \\ & \Phi_q^\dagger\Phi_q\end{pmatrix}\otimes\sigma^l\otimes\mathbf{1}_3 - i\begin{pmatrix} & \Phi_q W_j \\ W_j\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^k\otimes\mathbf{1}_3$$
$$+i\begin{pmatrix} & \Phi_q W_k \\ W_k\Phi_q^\dagger & \end{pmatrix}\otimes\sigma^j\otimes\mathbf{1}_3 + i\begin{pmatrix} & \sigma^3\Phi_q \\ \Phi_q^\dagger\sigma^3 & \end{pmatrix}\otimes(A_j\sigma^k - A_k\sigma^j)\otimes\mathbf{1}_3, \tag{B.18}$$

and

$$\begin{aligned}\kappa^{-1}\mathrm{Tr}(\Omega_{jk}\Omega_{jk}) = &-\frac{44}{3}F_{jk}F_{jk} - 6\mathrm{Tr}(W_{jk}W_{jk}) - 8\mathrm{Tr}(G_{jk}G_{jk}) - 288\mathrm{Tr}(\Phi_q^\dagger\Phi_q)^2 \\ &- 48\mathrm{Tr}\left[(\partial_j\Phi_q)^\dagger(\partial_j\Phi_q)\right] - 48\mathrm{Tr}(W_j^2\Phi_q^\dagger\Phi_q) - 48A_j^2\mathrm{Tr}(\Phi_q^\dagger\Phi_q) \\ &+ 96A_j\mathrm{Tr}(\Phi_q W_j\Phi_q^\dagger\sigma^3) - 48i\mathrm{Tr}\left[\left((\partial_j\Phi_q)^\dagger\Phi_q - \Phi_q^\dagger(\partial_j\Phi_q)\right)W_j\right] \\ &+ 48iA_j\mathrm{Tr}\left[\left(\Phi_q(\partial_j\Phi_q)^\dagger - (\partial_j\Phi_q)\Phi_q^\dagger\right)\sigma^3\right].\end{aligned} \tag{B.19}$$

Therefore

$$\begin{aligned}\kappa^{-1}\mathrm{Tr}(\Omega^2) = &-2\left(\frac{22}{3}F^2 + 3\mathrm{Tr}(W^2) + 4\mathrm{Tr}(G^2)\right) - 576\mathrm{Tr}(\Phi_q^\dagger\Phi_q)^2 \\ &- 72\mathrm{Tr}\left[(\partial_\mu\Phi_q)^\dagger(\partial_\mu\Phi_q)\right] - 72\mathrm{Tr}\left(W_\mu W_\mu\Phi_q^\dagger\Phi_q\right) - 72A_\mu A_\mu\mathrm{Tr}\left(\Phi_q^\dagger\Phi_q\right) \\ &+ 144A_\mu\mathrm{Tr}\left(\Phi_q^\dagger\sigma^3\Phi_q W_\mu\right) - 72iA_\mu\mathrm{Tr}\left[\left((\partial_\mu\Phi_q)\Phi_q^\dagger - \Phi_q(\partial_\mu\Phi_q)^\dagger\right)\sigma^3\right] \\ &+ 72i\mathrm{Tr}\left[\left(\Phi_q^\dagger(\partial_\mu\Phi_q) - (\partial_\mu\Phi_q)^\dagger\Phi_q\right)W_\mu\right].\end{aligned} \tag{B.20}$$

Expressing the coefficients a_2 and a_4 as in the section 3.1 we can further write

$$a_2 = \frac{3\kappa}{4\pi^2}(|\Upsilon_e|^2 + |\Upsilon_\nu|^2)\int d^4x 3|H|^2, \tag{B.21}$$

and

$$\begin{aligned}a_4 = \frac{\kappa}{8\pi^2}\int d^4x\Big[&(3|\Upsilon_\nu|^4 + 3|\Upsilon_e|^4)|H|^4 - (3|\Upsilon_\nu|^2 + 3|\Upsilon_e|^2)\mathrm{Tr}|D_\mu H|^2 \\ &+ \frac{11}{9}F^2 + \frac{1}{2}\mathrm{Tr}(W^2) + \frac{2}{3}\mathrm{Tr}(G^2) + 3\varepsilon^{jkl}F_{jk}F_{0l} - \frac{3}{2}\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l})\Big].\end{aligned} \tag{B.22}$$

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2.1.3 Outlook

The proposed framework for the description of the Standard Model of particle physics seems to be promising. First of all, it is, from the very construction, free of the old fermion doubling problem. Furthermore, its formulation already has the Lorentzian symmetry included. The interplay between the fermion doubling and the Lorentzian structure is also clearly visible. Moreover, the Dirac operator is chiral and this is the feature that seems to be to of crucial importance. It is worth stressing that the chirality was also a potential source of the fermion doubling problem in lattice gauge theories as well as within the almost-commutative framework [127]. The non-product geometry sheds new light on these old problems. The obtained results suggest that this framework may be the right one for the description of models present in particle physics. The story is not finished yet. The intriguing feature that the restriction of the full triple to constant functions gives a Riemannian triple without a real structure, while the restriction to the algebra of functions on the Minkowski spacetime produces real even Lorentzian spectral triple, requires further investigations. The natural continuation of this path of research is the rigorous mathematical understanding of these types of non-product geometries.

Furthermore, the nontrivial relation to twisted spectral triples has to be investigated in the future, especially the twist by a pseudo-Riemannian structure is an intriguing subject for further research. It seems reasonable to claim that there is also a deeper relationship between our model and the approach investigated in [130]. All these aspects hopefully will shed new light on the geometry of the Standard Model, and may also be used for the geometric formulation of theories going beyond the Standard Model.

2.2 Non-product geometry for modified gravity theories

The fact that the classical General Relativity can be described in terms of spectral data associated with a given spacetime motivates the search for an analogous formulation of models that could describe certain types of *modified gravity theories*. Since, due to Connes' reconstruction theorem, there is no place for such models within the class of commutative spectral triples, we have to go one step further. However, also almost-commutative framework does not allow for such modifications since the Dirac operator in the product geometry is of the form that fixes the continuous part to be the one for a classical geometry. Therefore, non-product types of geometry appear as a possible solution.

The first candidate for a model that potentially could be derived from the spectral data is the bimetric theory of gravity, which has its origin in ghost-free Hassan-Rosen theory [135, 136, 137].

The action of this model reads

$$
\begin{aligned}
S = & -\frac{M_g^2}{2}\int_{\mathcal{M}} \sqrt{-\det\, g}\, R(g) - \frac{M_f^2}{2}\int_{\mathcal{M}} + m^2 M_g^2 \int_{\mathcal{M}} \sqrt{-\det\, g}\, \sum_{n=0}^{4} \beta_n e_n(\sqrt{g^{-1}f}) \\
& + \int_{\mathcal{M}} \sqrt{-\det\, g}\, \mathcal{L}_m(g,\Phi) + \alpha \int_{\mathcal{M}} \sqrt{-\det\, g}\, \mathcal{L}_m(f,\Phi),
\end{aligned} \tag{14}
$$

where g and f are two metric tensors with $R(g)$ and $R(f)$ being their Ricci scalars, respectively. The two mass scales M_g and M_f are present, together with other free parameters: $\beta_0, \ldots, \beta_4, m^2, \alpha$. The potential term is constructed out of elementary symmetric polynomials e_n in eigenvalues of the matrix $\sqrt{g^{-1}f}$. For each metric there is a respective interaction term $\mathcal{L}_m$ with a matter field Φ.

This model was intensively studied both analytically and numerically [138, 139, 140]. It was demonstrated that in the case when the two metrics g and f are of the

Friedmann-Lemaître-Robertson-Walker (FLRW) type, this model can, for a certain range of parameters, reconstruct the standard ΛCDM model and, in addition, is a theory that potentially could describe the dark matter sector. Several cosmological scenarios were also analysed.

The lack of geometric interpretation together with potential applications of this model were the reasons for choosing it as a first candidate for the search of its spectral formulation. Motivated by the results from [141], where a certain type of non-product noncommutative geometry was introduced, we extented this analysis and search for a connection to bimetric theories. The model introduced in [141] differs from the almost-commutative type of geometries by the choice of the Dirac operator. The continuous part of the product Dirac operator (9) is replaced by

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \tag{15}$$

where D_1 and D_2 are two independent Dirac operators on $\mathcal{M}$. The spectral action for flat FLRW type geometries was computed therein, and the analysis of certain cosmological solutions was performed.

2.2.1 Doubled geometries - dynamical stability of solutions for FLRW models

We generalize the result from [141] into arbitrary FLRW geometries, and find relations between this type of non-product models, to which we refer as either *doubled geometry* or *two-sheeted models*, and bimetric gravity theories. The choice of FLRW geometries is motivated by the aforementioned existing numerical analysis performed for bimetric gravity models as well as by the fact that FLRW geometries were analysed in the classical case (i.e. for one copy of a manifold only) from the spectral perspective [142, 143, 144] and certain limits can be performed as cross-checks for the obtained new results.

First, using the method based on Wodzicki residue, we compute the spectral action for models which are flat or have either positive or negative curvature, and, in contrary to [141], their lapse functions are nontrivial. In all the cases, the interaction potential is, as expected, of the same form and possesses features that are characteristic to bimetric gravity models. Mainly, the potential term can be represented in the form $\mathbb{V}\left(\sqrt{g_2^{-1}g_1}\right)\sqrt{\det\, g_2}$, where $\mathbb{V}$ is a certain function which depends only on the (eigenvalues of the) matrix $\sqrt{g_2^{-1}g_1}$ constructed out of the two FLRW metrics g_1, g_2 present in the model. Furthermore,

$$\mathbb{V}\left(\sqrt{g_2^{-1}g_1}\right)\sqrt{\det\, g_2} = \mathbb{V}\left(\sqrt{g_1^{-1}g_2}\right)\sqrt{\det\, g_1}, \tag{16}$$

exactly as in the case of bimetric models. However, the exact form of this interaction term is different. Instead of a polynomial, we have here a rational function which can be expressed as a ratio of certain combinations of elementary symmetric polynomials.

Having derived the effective Lagrangian for the doubled geometry model and knowing its basic symmetries, we then proceed further with the analysis of its dynamics. We first derive the equations of motion and then discuss their coupling with stress-energy tensor, i.e. the interaction with matter. Based on microscopic considerations we argue the correctness of this procedure. Since we are interested in analysing cosmological scenarios, we take the stress-energy tensor for a perfect fluid. These choices are not the most general ones but cover most of the scenarios we are interested in.

Unfortunately, even for this simplified version, the solutions of the equations of motion cannot be easily found. Instead of developing numerical tools to get around this problem, we decide to first check if the classical solutions of the Einstein equations obtained for each sheet separately are stable under infinitesimal perturbations within our doubled geometry framework.

We derive a system of differential equations for these perturbations and solve it for a series of different cosmological scenarios. For each of these cases, there is always a range of parameters for which the stability of the solutions is found. We then discuss the possible cosmological implications of this model.

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Stability of Friedmann-Lemaître-Robertson-Walker solutions in doubled geometries.

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Motivated by the models of geometry with discrete spaces as additional dimensions we investigate the stability of cosmological solutions in models with two metrics of the Friedmann-Lemaître-Robertson-Walker type. We propose an effective gravity action that couples the two metrics in a similar manner as in bimetric theory of gravity and analyse whether standard solutions with identical metrics are stable under small perturbations.

I. INTRODUCTION

The spectacular success of geometry in the description of large-scale structure of the Universe (general relativity) as well as fundamental interactions (gauge theories) is one of the biggest achievements of modern physics. Yet the link between these two is still a major challenge to our understanding of the world. Apart from that there are multiple efforts to solve the puzzle of dark matter with interesting attempt to modify gravity. The bimetric theory [1], being one of consistently formulated models, appears to be a good candidate to solve the puzzle in accordance with the cosmological data [2–5]. However, the necessity to add a second metriclike field appears to be rather inelegant and is not well founded from the point of view of Riemannian geometry with the interaction potential between the two metrics introduced *ad hoc*, despite being motivated by non-linear generalizations of Fierz-Pauli massive gravity [6] which do not suffer from Boulware-Deser ghost problem [1, 7].

Surprisingly, the hint of a geometric explanation might come from models used in particle physics. In a quest to explain the structure of the Standard Model, a purely geometric interpretation of its content was proposed by Alain Connes using the tools of noncommutative geometry [8–10]. Taken seriously, it explains the existence of different fermions and gauge interactions as related to geometry of a finite type, related to a finite-dimensional algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with the derivation of the action linked to a general principle of Euclidean spectral action, which provides all terms, including the Yang-Mills-Higgs leading to the spontaneous symmetry breaking as well the pure gravity Einstein-Hilbert action.

A simplified model of this type, which was the first considered [11] in the early days of the development of the theory, describes a product of the smooth geometry (a four-dimensional manifold) with a two-point space. Such two-sheeted geometry, with a product structure is tractable in noncommutative geometry leading to a simple Yang-Mills-Higgs toy model. However, from the point of view of gravity an interesting question is whether it is admissible to have different metrics on the two separate sheets of this geometry. This question is a challenge not only from the conceptual but also from the technical point of view, as it requires the computation of the spectral action in a much more general case than the product geometry. In particular, the first question posed is whether the two metrics interact with each other. A first step in this direction was done in [12], where a simple model of two Friedmann-Lemaître-Robertson-Walker-type, flat geometries with identical lapse function was considered, resulting in the effective potential term linking the two geometries.

The present paper goes well beyond the restricted situation of the previous analysis, providing a full derivation of the potential linking the metric and the equations of motion as well as the analysis of their stability. Though our model differs significantly from the typical bimetric theory (none of the metrics can be thought of a background metric) the obtained potential is much similar to the bimetric case (though it is expressed as a rational function and not a polynomial in the eigenvalues of the metrics ratio). Moreover, the symmetric coupling to the matter and radiation makes it closer to the symmetric bimetric theory, where both metrics couple (in the same way) to matter and radiation.

The paper is organized as follows: we present the assumptions of our model (the structure of the two-sheeted geometry) and the methods of deriving the leading two terms of the spectral action using the pseudodifferential calculus and the Wodzicki residue. After computing the Euclidean action functional for flat as well for curved geometries we perform the Wick rotation and obtain a set of nonlinear differential equations for the four functions that describe the model. In the rest we focus on the stability of the symmetric solutions, which are the standard Friedmann-Lemaître-Robertson-Walker geometries for both sheets and analyse small perturbations for the typical cosmological solutions of dark-energy, matter and radiation-dominated universes. In the last section we briefly discuss the possible physical consequences and argue why the model is physically viable.

II. ALMOST COMMUTATIVE FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER MODELS

A. Almost-commutative geometries

The Gelfand-Naimark equivalence between topological spaces and commutative C^*-algebras was further enriched by A. Connes in order to include noncommutative algebras and also to describe more than only the topology. In his formulation of noncommutative geometry [13] the crucial role is played by a spectral triple which is a system $(\mathcal{A}, H, D)$ consisting of an unital $*$-algebra $\mathcal{A}$, Hilbert space H and a Dirac type selfadjoint operator acting on H. Usually, more additional structure is assumed (e.g. the existence of grading-type operator γ, and an anti-unitary operator J, called real structure) and further compatibility conditions between all these elements. The canonical commutative example is $(C^\infty(M), L^2(M,S), D_M)$, where M is a manifold equipped with a spin structure, $L^2(M,S)$ is the Hilbert space of square-integrable spinors, and $D_M = i\gamma^\mu(\partial_\mu + \omega_\mu)$ is the canonical Dirac operator expressed in the terms of the connection ω_μ on the spinor bundle.

It turns out that crucial from the applications in particle physics point of view are triples with algebras being tensor products of the above one which some finite-dimensional matrix algebras A_F. The Hilbert space is the tensor product of $L^2(M,S)$ with some finite-dimensional Hilbert space H_F on which A_F is represented, and its dimension determines the number of fermionic degrees of freedom in the theory. Grading operators and real structures are also composed in an appropriate way in order to define analogous objects on the resulting triple. The Dirac operator, however, is not just the simple tensor product of D_M and D_F, but has the following form:

$$D = D_M \otimes 1 + \gamma_M \otimes D_F. \tag{II.1}$$

The resulting triple forms the so-called almost-commutative geometry and have been the backbone of multiple models applied to the physics of elementary particles (see [14, 15]). The starting point to consider physical models based on spectral triples is the spectral action.

Its bosonic part is given by

$$\mathcal{S}(D) = \mathrm{Tr} f\left(\frac{D}{\Lambda}\right), \tag{II.2}$$

where Λ is some cut-off parameter and f is some smooth approximation of the characteristic function of the interval $[0, 1]$. In the case of particle physics models it reproduces the bosonic part of the Lagrangian of such theories minimally coupled to gravity, together with the standard Hilbert-Einstein action for the metric.

B. The classical geometry

We consider geometries described by the generalized Friedmann-Lemaître-Robertson-Walker metric,

$$ds^2 = b(t)^2 dt^2 + a(t)^2 \left(d\chi^2 + S_k^2(\chi)\left(d\theta^2 + \sin^2(\theta) d\phi^2\right)\right), \tag{II.3}$$

where

$$S_k(\chi) = \begin{cases} \sin(\chi), & k = 1, \\ \chi, & k = 0, \\ \sinh(\chi), & k = -1 \end{cases} \tag{II.4}$$

and $a(t), b(t)$ are positive (sufficiently smooth) functions.

The orthogonal coframe $\{\theta^a\}$ for ds^2 is defined so that $ds^2 = \theta^a\theta^a$. It allows us to immediately compute the spin connection ω which is determined by $d\theta^a = \omega^{ab} \wedge \theta^b$. Then, the Dirac operator is, in a local coordinates, given by

$$D = \gamma^a dx^\mu(\theta_a)\frac{\partial}{\partial x^\mu} + \frac{1}{4}\gamma^c \omega_{cab}\gamma^a\gamma^b, \tag{II.5}$$

where γ^a's are gamma matrices chosen to be antihermitian and so that $\gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab} I$.

Instead of the original Dirac operator we can equivalently analyse the operator, which is conformally rescaled, $D_h = h^{-1}Dh$, with the scale factor $h(t) = a(t)^{-3/2}b(t)^{-1/2}$. This assures that we can work with the Hilbert space of spinors, where the scalar product does not depend on $a(t)$ and $b(t)$.

C. The two-sheet almost commutative model

We consider a generalised almost-commutative geometry, which is described by a product-like spectral triple of the spectral triple over the manifold with the Friedmann-Lemaître-Robertson-Walker metric and the triple over two points. However, instead of the usual product Dirac operator, we take a more general one,

$$\mathcal{D} = \begin{pmatrix} D_1 & \gamma\Phi \\ \gamma\Phi^* & D_2 \end{pmatrix}, \tag{II.6}$$

where D_1, D_2 are both of the form (II.5), yet with possibly different scaling functions a and b, and Φ being a priori a field (which can be later restricted to be constant).

The choice of the full Dirac operator with the γ in the off-diagonal part is motivated by the fact that in the case of $D_1 = D_2$ it yields a usual almost-commutative product geometry. Note that, in principle one can study generalized objects with arbitrary order-zero operators on the off-diagonal of $\mathcal{D}$, so the only thing we require of γ is that it anticommutes with γ^a matrices and is not necessarily the chirality grading operator of the Euclidean spin geometry of the manifold M. In order to have the full Dirac operator hermitian we must require that γ is hermitian and, consequently, we have to normalize $\gamma^2 = 1$. However, we shall relax this assumption and consider also models with $\gamma^2 = -1$. This allows for much more flexibility, in particular, for the models that are derived from higher-dimensional Kaluza-Klein type geometries and would lead to some more realistic effective physical situations. One of the interesting possibilities is that when passing to the Lorentzian signature for the manifold M we can as well choose the Lorentzian signature for the discrete degrees of freedom. This possibility has been discussed for finite geometries, albeit in a different context of the Standard Model in [16], in the natural language of Krein spaces. What is important for our consideration is that the only difference will be that the operator $\mathcal{D}$ will be only Krein self-adjoint, meaning that γ will be antiselfadjoint and $\gamma^2 = -1$. To accommodate for both possibilities in the discrete degrees of freedom we do not fix γ^2 and we allow that (after normalization) $\gamma^2 = \kappa = \pm 1$.

To simplify the presentation in the paper we introduce the following matrices:

$$B(t) = \begin{pmatrix} \frac{1}{b_1(t)} & \\ & \frac{1}{b_2(t)} \end{pmatrix}, \quad A(t) = \begin{pmatrix} \frac{1}{a_1(t)} & \\ & \frac{1}{a_2(t)} \end{pmatrix}, \quad F(t,x) = \begin{pmatrix} & \Phi(t,x) \\ \Phi(t,x)^* & \end{pmatrix}. \quad \text{(II.7)}$$

D. The spectral action

For the geometry described by a given Dirac operator D the main object of interest is the Laplace-type operator D^2, which is a second-order differential operator acting on the sections of the doubled spinor bundle. Its symbol $\sigma_{D^2}(x,\xi)$ consists of three parts $\mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{a}_2$, each of $\mathfrak{a}_k(x,\xi)$ being homogeneous of degree k in ξ's. Then we compute the symbol of its inverse,

$$\sigma_{D^{-2}}(\xi) = \mathfrak{b}_0 + \mathfrak{b}_1 + \mathfrak{b}_2 + ..., \quad \text{(II.8)}$$

where $\mathfrak{b}_k(x,\xi)$ is homogeneous of order $-2-k$ in ξ (we briefly review the mathematical details of how the computations of the symbols are performed in the Appendix A) and use it to compute the first two terms of the spectral action for the considered model.

It can be expressed in terms of Wodzicki residua [17, 18] as,

$$\mathcal{S}(D) = \Lambda^4 \operatorname{Wres}(D^{-4}) + c\Lambda^2 \operatorname{Wres}(D^{-2}) = \int_M \int_{\|\xi\|=1} \left(\Lambda^4 \operatorname{Tr} \operatorname{Tr}_{Cl} \mathfrak{b}_0^2 + c\Lambda^2 \operatorname{Tr} \operatorname{Tr}_{Cl} \mathfrak{b}_2\right), \quad \text{(II.9)}$$

where Tr_{Cl} denotes the trace performed over the Clifford algebra and Tr is the trace over the matrices $M_2(\mathbb{C})$ that are used in the mild noncommutativity introduced in the model.

E. Flat geometries.

Although the topology of the flat case in physics is not exactly toroidal, from the point of view of local behaviour it is identical to such, which was already analysed for $b = 1$ in

[12]. In this section we generalize those results to the case with arbitrary function $b(t)$, so we consider here toroidal Friedmann-Lemaître-Robertson-Walker geometries described by the following metric in the coordinate system $(t,x)=(t,x^1,x^2,x^3)$:

$$ds^2 = b(t)^2dt^2 + a(t)^2\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right). \tag{II.10}$$

Hence an orthogonal frame for ds^2 is of the form,

$$\theta^0 = b(t)dt, \qquad \theta^1 = a(t)dx^1, \qquad \theta^2 = a(t)dx^2, \qquad \theta^3 = a(t)dx^3, \tag{II.11}$$

while the matrix of connection 1-forms is,

$$\omega = \frac{1}{a(t)b(t)}\begin{pmatrix} 0 & -(\partial_t a)\theta^1 & -(\partial_t a)\theta^2 & -(\partial_t a)\theta^3 \\ (\partial_t a)\theta^1 & 0 & 0 & 0 \\ (\partial_t a)\theta^2 & 0 & 0 & 0 \\ (\partial_t a)\theta^3 & 0 & 0 & 0 \end{pmatrix}. \tag{II.12}$$

As a result, the (single) Dirac operator takes the following form,

$$D = \frac{1}{b(t)}\gamma^0\left(\partial_t + \frac{3\partial_t a}{2a(t)}\right) + \frac{1}{a(t)}\gamma^j\partial_j, \tag{II.13}$$

and after the conformal rescaling $h(t) = a(t)^{-3/2}b(t)^{-1/2}$ we get

$$D_h = \frac{1}{b(t)}\gamma^0\left(\partial_t - \frac{\partial_t b}{2b(t)}\right) + \frac{1}{a(t)}\left(\gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right), \tag{II.14}$$

so that the full Dirac operator acting on the doubled Hilbert space of spinors is,

$$\mathcal{D} = \gamma^0\left(B(t)\partial_t - \partial_t B\right) + A(t)\gamma^j\partial_j + \gamma F(t,x). \tag{II.15}$$

The resulting Laplace-type operator in this model is of the following form:

$$\begin{aligned}\mathcal{D}^2 = &- B^2\partial_t^2 - A^2\partial^2 + B(\partial_t A)\gamma^0\gamma^k\partial_k + [F,A]\gamma\gamma^k\partial_k \\ &+ [F,B]\gamma\gamma^0\partial_t + \kappa F^2 + \gamma^0\gamma B(\partial_t F) + \gamma^j\gamma A(\partial_j F) \\ &+ \gamma^0\gamma[F,\partial_t B] + B(\partial_t^2 B) + B(\partial_t B)\partial_t - (\partial_t B)^2.\end{aligned} \tag{II.16}$$

The symbol $\sigma(\mathcal{D}^2) = \mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{a}_2$ is given by

$$\begin{aligned}\mathfrak{a}_2 &= B^2\xi_0^2 + A^2\xi^2, \\ \mathfrak{a}_1 &= i\left[B(\partial_t A)\gamma^0\gamma^k\xi_k + B(\partial_t B)\xi_0 + [F,A]\gamma\gamma^k\xi_k + [F,B]\gamma\gamma^0\xi_0\right], \\ \mathfrak{a}_0 &= \kappa F^2 - \gamma\gamma^0\left(B(\partial_t F) + [F,\partial_t B]\right) - \gamma\gamma^j A(\partial_j F) + B(\partial_t^2 B) - (\partial_t B)^2,\end{aligned} \tag{II.17}$$

where we denoted by $\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. Now, computing the symbol of $\mathcal{D}^{-2}$ using the prescription presented in Appendix A, we obtain $\mathfrak{b}_0(\mathcal{D})$ and $\mathfrak{b}_2(\mathcal{D})$. Then, taking the trace over the Clifford algebra and the matrices $M_2(\mathbb{C})$, and integrating over the cosphere bundle

$|\xi|^2 = 1$, we compute the Wodzicki residue that gives us the Euclidean spectral action of the considered model. The final result is,

$$\begin{aligned}\mathcal{S}(\mathcal{D}) \sim \int dt \Big\{ & \Lambda^4(a_1^3 b_1 + a_2^3 b_2) - \frac{c\Lambda^2}{12}\left(a_1^3 b_1 R(a_1,b_1) + a_2^3 b_2 R(a_2,b_2)\right) \\ & + c\kappa\Lambda^2|\Phi|^2 b_1 b_2 \frac{(a_1-a_2)^2}{(a_1b_2+a_2b_1)^2}\left[a_1^2(2a_2b_1+a_1b_2)+a_2^2(2a_1b_2+a_2b_1)\right] \\ & + c\kappa\Lambda^2|\Phi|^2\frac{(b_1-b_2)^2}{(a_2b_1+a_1b_2)^2}a_1^2a_2^2(a_1b_1+a_2b_2) - c\kappa\Lambda^2|\Phi|^2(a_1^3b_1+a_2^3b_2),\end{aligned} \tag{II.18}$$

where the scalar curvature for the flat spatial geometry is,

$$R(a,b) = 6\left(\frac{\partial_t a\,\partial_t b}{ab^3} - \frac{(\partial_t a)^2}{a^2b^2} - \frac{\partial_t^2 a}{ab^2}\right). \tag{II.19}$$

F. The non-flat case

In this subsection we concentrate on the case with positive ($k = 1$) curvatures, with the negative ($k = -1$) case that can be treated in a similar manner. Although the effective Lagrangian and the equations of motion are local and hence the dynamical terms are expected to be unchanged, we derive them explicitly using appropriate coordinates. For the case of $k = 1$ we use the spherical coordinates (t, χ, θ, ϕ), so that the metric is then described by:

$$ds^2 = b(t)^2dt^2 + a(t)^2\left(d\chi^2 + \sin^2(\chi)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right). \tag{II.20}$$

The orthogonal frame is given by

$$\theta^0 = b(t)\,dt, \quad \theta^1 = a(t)\,d\chi, \quad \theta^2 = a(t)\sin\chi\,d\theta, \quad \theta^3 = a(t)\sin\chi\sin\theta\,d\phi, \tag{II.21}$$

hence

$$\begin{aligned} & d\theta^0 = 0, \qquad d\theta^1 = \frac{\partial_t a}{ab}\theta^0\wedge\theta^1, \qquad d\theta^2 = \frac{\partial_t a}{ab}\theta^0\wedge\theta^2 + \frac{\cot\chi}{a}\theta^1\wedge\theta^2 \\ & d\theta^3 = \frac{\partial_t a}{ab}\theta^0\wedge\theta^3 + \frac{\cot\chi}{a}\theta^1\wedge\theta^3 + \frac{\cot\theta}{a\sin\chi}\theta^2\wedge\theta^3. \end{aligned} \tag{II.22}$$

Therefore the only nonvanishing components for the spin connection ω are [19, 20]:

$$\omega_{101} = \omega_{202} = \omega_{303} = \frac{\partial_t a}{ab}, \quad \omega_{212} = \omega_{313} = \frac{\cot\chi}{a}, \qquad \omega_{323} = \frac{\cot\theta}{a\sin\chi}. \tag{II.23}$$

Now, for the Dirac operator we get explicitly

$$D = \gamma^0\frac{1}{b}\left(\frac{\partial}{\partial t} + \frac{3}{2}\frac{\partial_t a}{a}\right) + \frac{1}{a}D_3, \tag{II.24}$$

where in this case

$$D_3 = \gamma^1\frac{\partial}{\partial\chi} + \gamma^2\csc\chi\frac{\partial}{\partial\theta} + \gamma^3\csc\chi\,\csc\theta\frac{\partial}{\partial\phi} + \gamma^1\cot\chi + \frac{1}{2}\gamma^2\cot\theta\,\csc\chi. \tag{II.25}$$

After the conformal rescaling by using $h(t)=a(t)^{-3/2}b(t)^{-1/2}$ we end up with the following Dirac operator

$$D_h=\frac{1}{b}\gamma^0\left(\frac{\partial}{\partial t}-\frac{\partial_t b}{b}\right)+\frac{1}{a}D_3, \tag{II.26}$$

Therefore, for the doubled model that we are considering, the Dirac operator is

$$\mathcal{D}=\gamma^0\left(B(t)\partial_t-\partial_t B\right)+A(t)D_3+\gamma F(t,x). \tag{II.27}$$

As a result we have

$$\begin{aligned}\mathcal{D}^2=&-B^2\partial_t^2+A^2D_3^2+B(\partial_t A)\gamma^0 D_3+[F,A]\gamma D_3\\&+[F,B]\gamma\gamma^0\partial_t+\kappa F^2+\gamma^0\gamma B(\partial_t F)-\gamma A(D_3F)\\&+\gamma^0\gamma[F,\partial_t B]+B(\partial_t^2 B)+B(\partial_t B)\partial_t-(\partial_t B)^2.\end{aligned} \tag{II.28}$$

In order to compute its symbol $\sigma_{\mathcal{D}^2}=\mathfrak{a}_2+\mathfrak{a}_1+\mathfrak{a}_0$ we first notice that the symbol of D_3^2 is given by:

$$\begin{aligned}\mathfrak{a}_2(D_3^2)\;=&\,\xi_\chi^2+\csc^2\chi\,\xi_\theta^2+\csc^2\chi\,\csc^2\theta\,\xi_\phi^2,\\ \mathfrak{a}_1(D_3^2)\;=&\,-i\left(2\cot\chi\,\xi_\chi+\cot\theta\,\csc^2\chi\,\xi_\theta+\gamma^1\gamma^2\cot\chi\,\csc\chi\,\xi_\theta+\right.\\&\left.+\gamma^1\gamma^3\csc\theta\,\cot\chi\,\csc\chi\,\xi_\phi+\gamma^2\gamma^3\cot\theta\,\csc\theta\,\csc^2\chi\,\xi_\phi\right),\\ \mathfrak{a}_0(D_3^2)\;=&\,-\frac{1}{2}\gamma^1\gamma^2\cot\theta\,\cot\chi\,\csc\chi+\csc^2\chi-\cot^2\chi\\&+\frac{1}{2}\csc^2\theta\,\csc^2\chi-\frac{1}{4}\cot^2\theta\,\csc^2\chi.\end{aligned} \tag{II.29}$$

As a result, for the operator $\mathcal{D}^2$, we have

$$\begin{aligned}\mathfrak{a}_2\;=&\,B^2\xi_0^2+A^2\xi_\chi^2+\csc^2\chi\,A^2\xi_\theta^2+\csc^2\chi\,\csc^2\theta\,A^2\xi_\phi^2,\\ \mathfrak{a}_1\;=&\,-i\left\{2\cot\chi\,A^2\xi_\chi+\cot\theta\,\csc^2\chi\,A^2\xi_\theta-B(\partial_t B)\xi_0+\right.\\&+\gamma^1\gamma^2A^2\cot\chi\,\csc\chi\,\xi_\theta+\gamma^1\gamma^3A^2\csc\theta\,\cot\chi\,\csc\chi\,\xi_\phi\\&+\gamma^2\gamma^3A^2\cot\theta\,\csc\theta\,\csc^2\chi\,\xi_\phi-B(\partial_t A)\gamma^0\gamma^1\xi_\chi\\&-B(\partial_t A)\gamma^0\gamma^2\csc\chi\,\xi_\theta-B(\partial_t A)\gamma^0\gamma^3\csc\chi\,\csc\theta\,\xi_\phi-\\&-[F,A]\gamma\gamma^1\xi_\chi-[F,A]\gamma\gamma^2\csc\chi\,\xi_\theta-[F,A]\gamma\gamma^3\csc\chi\,\csc\theta\,\xi_\phi\\&\left.-[F,B]\gamma\gamma^0\xi_0\right\},\\ \mathfrak{a}_0\;=&\,A^2\left(\csc^2\chi-\cot^2\chi+\frac{1}{2}\csc^2\theta\,\csc^2\chi-\frac{1}{4}\cot^2\theta\,\csc^2\chi\right)\\&+\kappa F^2+B(\partial_t^2B)-(\partial_t B)^2-\frac{1}{2}\gamma^1\gamma^2\cot\theta\,\cot\chi\,\csc\chi\\&+B(\partial_t A)\gamma^0\gamma^1\cot\chi+\frac{1}{2}B(\partial_t A)\gamma^0\gamma^2\cot\theta\,\csc\chi\\&+[F,A]\gamma\gamma^1\cot\chi+\frac{1}{2}[F,A]\gamma\gamma^2\cot\theta\,\csc\chi+\gamma^0\gamma[F,\partial_t F]\\&-\gamma\gamma^0B(\partial_t F)-\gamma\gamma^1A\left(\partial_\chi F+F\cot\chi\right)-\gamma\gamma^2A\csc\chi\\&+\left(\partial_\theta F+\frac{F}{2}\cot\theta\right)-\gamma\gamma^3A\csc\chi\,\csc\theta\,\partial_\phi F.\end{aligned}$$

Using the prescription presented in Appendix A, we first compute the symbols $\sigma_{\mathcal{D}^{-2}} = \mathfrak{b}_0 + \mathfrak{b}_1 + \mathfrak{b}_2 + ...$, then we proceed, in an exactly similar manner as in the case of the toroidal geometry, to compute the spectral action. The result is,

$$\begin{aligned}\mathcal{S}(\mathcal{D}) \sim \int dt \Big\{ &\left(\Lambda^4 - c\kappa\Lambda^2|\Phi|^2\right)\left(a_1^3 b_1 + a_2^3 b_2\right) - \frac{c\Lambda^2}{12}\left(a_1^3 b_1 R(a_1,b_1) + a_2^3 b_2 R(a_2,b_2)\right) \\ &+ c\kappa\Lambda^2|\Phi|^2 b_1 b_2 \frac{(a_1-a_2)^2}{(a_1b_2+a_2b_1)^2}\left[a_1^2(2a_2b_1+a_1b_2) + a_2^2(2a_1b_2+a_2b_1)\right] + \\ &+ c\kappa\Lambda^2|\Phi|^2 \frac{(b_1-b_2)^2}{(a_2b_1+a_1b_2)^2} a_1^2 a_2^2 (a_1b_1+a_2b_2)\Big\},\end{aligned} \tag{II.30}$$

span

where now $R(a, b)$ denotes the scalar of curvature for spherical spatial geometries,

$$R(a,b) = 6\left(\frac{\partial_t a\,\partial_t b}{ab^3} - \frac{(\partial_t a)^2}{a^2b^2} - \frac{\partial_t^2 a}{ab^2} + \frac{1}{a^2}\right). \tag{II.31}$$

Note that the action functional differs (II.30) from (II.18) only through the last term that arises from the scalar curvature of the spherical spatial geometry, where a relevant term depending on $k = 1$ is added. The above result can be generalized to the negative curvature case (we omit straightforward but tedious parametrization and computation of symbols). In fact, taking $R(a, b)$ as $R(a, b, k)$, depending on the space curvature k, we have a general action functional for all geometries in the doubled spacetime Friedmann-Lemaître-Robertson-Walker models.

G. The interactions of the metrics

Before we pass to the equations of motions an their stability, let us briefly compare the effective potential describing the interaction between the two metrics to the bimetric gravity models[1, 4, 21]. Certainly, apart from the fact that we have an action for two metrics, there is a much deeper symmetry between the two, since neither plays a role of a „background" metric. In fact, the usual solution in the case of vanishing α gives both metric totally independent of each other. Introducing the variables,

$$x = \frac{b_1}{b_2}, \qquad y = \frac{a_1}{a_2},$$

which depend only on the entries of the matrix $X^a_{\ c} = g_2^{ab} g_{1bc}$, we can express the interactions between the metrics as proportional to:

$$\mathbb{V}\left(\sqrt{g_2^{-1}g_1}\right)\sqrt{g_2} = \mathbb{V}\left(\sqrt{g_1^{-1}g_2}\right)\sqrt{g_1}, \tag{II.32}$$

where the function $\mathbb{V}\left(\sqrt{g_2^{-1}g_1}\right)$ is of the form:

$$\frac{x^2 + 2xy - 2x^2y + y^2 - 6xy^2 + 4x^2y^2 + 4xy^3 - 6x^2y^3 + x^3y^3 - 2xy^4 + 2x^2y^4 + xy^5}{(x+y)^2}$$

which can be efficiently expressed as a rational function of the symmetric polynomials in $\sqrt{X}$.

We stress that the resulting model possesses features that are characteristic to bimetric gravity models: the potential $\mathbb{V}$ depends on the metrics only through $\sqrt{X}$ and satisfies (II.32). On the other hand, in the usual bimetric models such potential is a polynomial in eigenvalues of $\sqrt{X}$ rather than a rational function. It was proposed in [22] that the construction presented here might results in the derivation of bimetric theories out of the geometric data. The above result suggest that indeed this class of models resembles some characteristics of bimetric gravity models, but is a different one. We postpone for the future research the detailed analysis of these differences and their cosmological implications.

H. The equations of motion

The action functional (II.30) depends on the field B only via b_1 and b_2 but not their derivatives. As a result, b_1 and b_2 are not dynamical and its Euler-Lagrange equations give rise to the constraints of the model. Moreover, due to the reparametrization invariance we can fix one of these functions or relate them with each other.

Furthermore, the action functional was derived for the Euclidean model and to pass to physical situation we need to perform Wick rotation, as described in the [12]. In our case, this will affect only the square of the time derivative of the scaling factors $a_i(t)$, which will change signs. Consequently, the action and the equation of motion for the rest of this paper are in the Lorentzian signature of the metric $(-,+,+,+)$. Let us remind that the discrete degrees of freedom of the geometry might be Riemannian or pseudo-Riemannian, which results in the appropriate choice of the sign κ.

After integration by parts and omitting the boundary terms that are full derivatives in t, we obtain the following action for the pure gravity Friedmann-Lemaître-Robertson-Walker doubled geometries for the Lorentzian signature and arbitrary spatial curvature k,

$$\begin{aligned}\mathcal{S}_k(\mathcal{D}) = \left(\frac{c\Lambda^2}{12}\right)\Bigg\{\int dt\, \Big(&\Lambda_e(a_1^3 b_1 + a_2^3 b_2) - 6k\,(a_1 b_1 + a_2 b_2) \\ &+ 6\left(\frac{a_1}{b_1}(\partial_t a_1)^2 + \frac{a_2}{b_2}(\partial_t a_2)^2\right) \\ &+ \alpha\, b_1 b_2 \frac{(a_1 - a_2)^2}{(a_1 b_2 + a_2 b_1)^2}\left[a_1^2(2a_2 b_1 + a_1 b_2) + a_2^2(2a_1 b_2 + a_2 b_1)\right] \\ &+\alpha\,\frac{(b_1 - b_2)^2}{(a_2 b_1 + a_1 b_2)^2} a_1^2 a_2^2 (a_1 b_1 + a_2 b_2)\Big)\Bigg\},\end{aligned} \tag{II.33}$$

where we have factored out the overall constant so that the dynamical term appears only with a numerical factor, denoted the effective cosmological constant by Λ_e and introduced the effective coupling between the two metrics by α:

$$\Lambda_e = \frac{12}{c}\left(\Lambda^2 - c\kappa|\Phi|^2\right), \qquad \alpha = 12|\Phi|^2\kappa.$$

Unlike the bare cut-off parameter Λ, here the effective cosmological constant can vanish or be negative for a particular model. We shall use the above convention with Λ_e and α throughout the rest of the paper.

The four Euler-Lagrange equations take the following form,

$$\Lambda_e = 6H_{b,i}^2 + 6\frac{k}{a_i^2} - \frac{\alpha}{a_i}V(a_i, a_{i'}, b_i, b_{i'}), \tag{II.34}$$

with

$$V(a_1, a_2, b_1, b_2) = a_1 + \frac{8a_1a_2(a_1^2 - a_2^2)b_2^3}{(a_2b_1 + a_1b_2)^3} + \frac{2a_2(a_2^2 + 2a_1a_2 - 5a_1^2)b_2^2}{(a_2b_1 + a_1b_2)^2},$$

and

$$12\frac{\partial_t^2 a_i}{a_i b_i^2} + 6H_{b,i}^2 - 3\Lambda_e + 6\frac{k}{a_i^2} - 12\frac{(\partial_t a_i)(\partial_t b_i)}{a_i b_i^3} - \alpha W(a_i, a_{i'}, b_i, b_{i'}) = 0, \tag{II.35}$$

with

$$\begin{aligned} W(a_1, a_2, b_1, b_2) =& \, 3 - 2\frac{a_2b_2(a_2^2 - 4a_1a_2 + 9a_1^2)}{a_1^2(a_2b_1 + a_1b_2)} + 2\frac{a_2b_2^2(11a_1^2 - 2a_1a_2 - 3a_2^2)}{a_1(a_2b_1 + a_1b_2)^2} \\ &- 8\frac{a_2b_2^3(a_1^2 - a_2^2)}{(a_2b_1 + a_1b_2)^3}. \end{aligned}$$

In the above equations we use the convention that $(i, i') = \{(1, 2), (2, 1)\}$, and $H_{b,j} = \frac{\partial_t a_j}{a_j b_j}$ are the generalized Hubble parameters. Before we analyse the inclusion of matter fields and possible solutions, let us observe that in the flat $k = 0$ case, inserting Λ_e from first two equations in last two, one obtains

$$\begin{aligned} &\frac{6}{b_1}\partial_t H_{b,1} + \alpha a_2 b_2 \frac{a_2b_1 - a_1b_2}{a_1^2} L(a_1, a_2, b_1, b_2) = 0, \\ &\frac{6}{b_2}\partial_t H_{b,2} + \alpha a_1 b_1 \frac{a_1b_2 - a_2b_1}{a_2^2} L(a_1, a_2, b_1, b_2) = 0, \end{aligned} \tag{II.36}$$

with some rational function $L(a_1, a_2, b_1, b_2)$, so in particular, whenever $a_1b_2 = a_2b_1$ both $H_{b,1}$ and $H_{b,2}$ must be constant.

III. INTERACTION WITH MATTER FIELDS AND RADIATION

The equation of motion derived in the previous section describe the empty universe in the doubled model. Here, we can ask how they are modified by the presence of the matter fields. The crucial point is to see how the effective matter and radiation action depend on the components of the metrics described in terms of fields a_1, a_2, b_1, b_2. The main difficulty is the passage from the microscopic action for spinor and gauge fields to the effective averaged energy-momentum tensor in the Einstein equations.

The microscopic action for the spinor fields in the doubled universe will be the usual fermionic action $\overline{\Psi}\mathcal{D}\Psi$. Since both components of the spinor couple to the respective Dirac operators D_1 and D_2 on each of the single sheets separately, and the Ψ field is, by assumption, independent of the metric fields, we conclude that the resulting action will be split into separate actions that do not mix the metric components on each of the single universes.

Similar argument can be used for the radiation energy-momentum tensor that originates from the gauge fields over the considered model. As the model has two $U(1)$ symmetries there are two gauge fields that couple to the Higgs field. A linear combination of them will become a massive one, due to spontaneous symmetry breaking of the Higgs field, whereas another linear combination will correspond to the massless photons. Again, the effective Yang-Mills action for the photon field will not mix the metric components over the two sheets and therefore we shall have independent tensor-energy components for each equation.

These heuristic arguments suggest that the effective equations of motion are modified by the respective components of the overall energy-momentum tensor $T^0_{\ 0}$ and $T^1_{\ 1}$, which depend separately on a_1, b_1 and a_2, b_2,

$$
\begin{aligned}
&6H_{b,i}^2 + \frac{6k}{a_i^2} - \Lambda_e - \frac{\alpha}{a_i} V(a_i, a_{i'}, b_i, b_{i'}) = -2T^0_{\ 0}(a_i, b_i),\\
&12\frac{\partial_t^2 a_i}{a_i b_i^2} + 6H_{b,i}^2 - 3\Lambda_e + \frac{6k}{a_i^2} - 12\frac{\partial_t a_i\, \partial_t b_i}{a_i b_i^3} - \alpha W(a_i, a_{i'}, b_i, b_{i'}) = -6T^1_{\ 1}(a_i, b_i),
\end{aligned}
\tag{III.1}
$$

for $(i, i') = \{(1,2), (2,1)\}$.

As in the conventional cosmology we consider the model of the perfect fluid, i.e. the stress-energy tensor is taken to be of the form

$$
T^g_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}, \tag{III.2}
$$

where P is refered to as pressure, while ρ is called energy density. For the generalized Friedmann-Lemaître-Robertson-Walker metric, the vector u^μ is $\left(\frac{1}{b(t)}, 0, 0, 0\right)$, so that $u_\mu u^\mu = -1$. As a result, $T^0_{\ 0} = -\rho$ and $T^1_{\ 1} = P$.

Furthermore, the continuity equation $\nabla_\mu T^{\mu\nu} = 0$ reduces to the standard one:

$$
\frac{\partial \rho}{\partial t} + 3(\rho + P)\frac{\partial_t a}{a} = 0. \tag{III.3}
$$

We assume that the thermodynamics of the matter content is characterized by the following equation of state:

$$
P(t) = w\rho(t). \tag{III.4}
$$

From the continuity equation we immediately infer that then

$$
\rho(t) = \eta\, a(t)^{-3(1+w)}, \tag{III.5}
$$

where η is the proportionality constant, exactly as in the standard cosmology.

The resulting Einstein equations for the double-sheeted universe are of the following form:

$$
\begin{aligned}
&6H_{b,i}^2 + \frac{6k}{a_i^2} - \Lambda_e - \frac{\alpha}{a_i} V(a_i, a_{i'}, b_i, b_{i'}) = \frac{2\eta}{a_i^{3(1+w)}}\\
&12\frac{\partial_t^2 a_i}{a_i b_i^2} + 6H_{b,i}^2 - 3\Lambda_e + \frac{6k}{a_i^2} - 12\frac{\partial_t a_i\, \partial_t b_i}{a_i b_i^3} - \alpha W(a_i, a_{i'}, b_i, b_{i'}) = -\frac{6w\eta}{a_i^{3(1+w)}},
\end{aligned}
\tag{III.6}
$$

for $(i,i') = \{(1,2),(2,1)\}$.

We stress that the above model is a straightforward generalization of the classical one for the doubled theory, with the only difference that we allow two different scaling factors and the interaction between them derived from the spectral action. Indeed, for $a_1 = a_2 \equiv a$, $b_1 = b_2 = 1$, or $\alpha = 0$, equations of motion reduces to the usual Friedmann equations yielding the well-known solutions.

In what follows we shall aim to analyse the possibility of small perturbations of the classical solutions of Friedmann-Lemaître-Robertson-Walker models, trying to answer the question whether the double-sheeted universe is stable. However, before we start the computations to see when the small perturbations of the classical solution are possible, let us observe that thanks to the reparametrisation invariance of the time variable in the equations (II.34), (II.35), we can decide to fix either b_1 or b_2 or relate them with each other. There are, in principle, many choices of the possible parametrizations and we choose a particular one, which is motivated by the existing symmetry with respect to the exchange between left and right modes in the geometric setup we consider. We shall set $b_1(t) + b_2(t) = 2$, therefore, effectively, one can introduce a new function,

$$b_1(t) = 1 + b(t), \qquad b_2(t) = 1 - b(t),$$

and derive the equations of motion for $a_1(t), a_2(t)$ and $b(t)$. Taken an appropriate linear combination of the derivatives of (II.34) and the equations (II.35) we shall obtain three nonlinear first order differential equations for these functions.

Despite the fact that a full analysis of these equations is complicated and can be done possibly only numerically, we can obtain some significant results.

We shall finish this section by a remark that one cannot a'priori assume that both lapse functions are identically 1. Indeed, we shall see that such solutions (in the linearised regime) are not possible. Moreover, a far more general argument holds also for the full equations in the case of the empty universe. Then, there are no solutions with $b(t) = 0$ apart from $a_1(t) = a_2(t)$. The argument is quite simple and relies on algebraic manipulation of the equations (II.34). Indeed, assuming $b_1(t) = 1 = b_2(t)$ and subtracting the equations we obtain the following relation between a_1 and a_2,

$$2\alpha \frac{a_1 - a_2}{(a_1 + a_2)^2} \left(a_1^2 + 4a_1a_2 + a_2^2\right) \left(a_2^2\, \partial_t a_1 + a_1^2\, \partial_t a_2\right) = 0,$$

which is true only if $a_1 = a_2$ (since both a_1, a_2 are positive functions) or $\frac{1}{a_1} + \frac{1}{a_2} = \text{const}$. The latter condition can be solved, and when used in either of the first two equations (II.34) it leads to the constant solutions for a_1 and a_2. Therefore, the functional based on the action (II.33) has extremal points only if the time scaling factor differs for the two metrics, so $b(t) \neq 0$. We leave aside the interpretation of this observation and its potential physical consequences to see whether the solutions that differ from the standard ones allow physically feasible models.

IV. PERTURBATIVE SOLUTIONS

In what follows we study infinitesimal perturbations of the classical solutions of the Einstein equations in different scenarios like empty universe with a cosmological constant, with and without curvature, the matter dominated (i.e. for $w = 0$) flat universe with a vanishing cosmological constant $\Lambda_e = 0$ and the radiation dominated (i.e. for $w = \frac{1}{3}$) flat universe.

Our working assumption is that we look for small perturbations around the symmetric, product, geometry of the form,

$$a_1(t) = a(t) + \epsilon r_1(t), \qquad a_2(t) = a(t) + \epsilon r_2(t), \qquad b(t) = \epsilon s(t). \tag{IV.1}$$

and linearise the equations of motion, taking the first terms in ϵ.

In the zeroth order, we obtain (from all equations, as expected):

$$6\,\frac{(\dot{a}(t))^2}{a(t)^2} - \Lambda_e + 6\,\frac{k}{a(t)^2} = 2\,\frac{\eta}{a(t)^{3+3w}}, \tag{IV.2}$$

whereas the first order yields the following set of linear equations for r_1, r_2 and s, for the function $a(t)$, which already satisfies the equation (IV.2):

$$\begin{aligned}
\dot{r}_1(t) &= \frac{3\lambda^2 a(t)^2(1+w) - (\dot{a}(t)^2 + k)(1+3w)}{2a(t)\dot{a}(t)} r_1(t) + \left(\dot{a}(t) + \alpha\frac{a(t)^2}{6\dot{a}(t)}\right) s(t), \\
\dot{r}_2(t) &= \frac{3\lambda^2 a(t)^2(1+w) - (\dot{a}(t)^2 + k)(1+3w)}{2a(t)\dot{a}(t)} r_2(t) - \left(\dot{a}(t) + \alpha\frac{a(t)^2}{6\dot{a}(t)}\right) s(t), \\
\dot{s}(t) &= \frac{3}{2}\frac{\dot{a}(t)}{a(t)}\left(\frac{r_1(t) - r_2(t)}{a(t)} - 2s(t)\right),
\end{aligned} \tag{IV.3}$$

where we have introduced $\Lambda_e = 6\lambda^2$ for simplicity, and denote the time derivative by a dot.

Note that for a given background solution $a(t)$ we have a homogeneous equation for the sum $r_1(t) + r_2(t)$, which has a simple solution that, however satisfies reasonable initial conditions $r_1(t_0) = r_2(t_0) = 0$ if and only if it is constantly 0. Therefore, we may freely restrict ourselves to the case $r_1(t) = r(t) = -r_2(t)$, and final set of perturbative equations,

$$\begin{aligned}
\dot{r}(t) &= \frac{3\lambda^2 a(t)^2(1+w) - (\dot{a}(t)^2 + k)(1+3w)}{2a(t)\dot{a}(t)} r(t) + \left(\dot{a}(t) + \alpha\frac{a(t)^2}{6\dot{a}(t)}\right) s(t), \\
\dot{s}(t) &= 3\frac{\dot{a}(t)}{a(t)}\left(\frac{r(t)}{a(t)} - s(t)\right).
\end{aligned} \tag{IV.4}$$

A. The empty universe

In the case of an empty, or dark-energy dominated universe, we have the simple case of $\eta = 0$ and cosmological solutions depending only on the curvature k and the cosmological constant Λ_e.

1. De Sitter universe ($k = 0$)

The solution of (IV.2) is,

$$a(t) = a_0 \exp\left(\sqrt{\frac{\Lambda_e}{6}}t\right), \tag{IV.5}$$

and the equations of motion for r, s are:

$$\begin{aligned} \dot{r}(t) &= \lambda r(t) + a(t)s(t)\left(\lambda + \frac{\alpha}{6\lambda}\right), \\ \dot{s}(t) &= 3\lambda\frac{r(t)}{a(t)} - 3\lambda s(t). \end{aligned} \tag{IV.6}$$

Solving this system of linear equations we obtain,

$$\begin{aligned} s(t) &= C_1 e^{-\frac{3}{2}\lambda t + \frac{1}{2}\sqrt{21\lambda^2+2\alpha}t} + C_2 e^{-\frac{3}{2}\lambda t - \frac{1}{2}\sqrt{21\lambda^2+2\alpha}t}, \\ r(t) &= C_3 e^{-\frac{1}{2}\lambda t + \frac{1}{2}\sqrt{21\lambda^2+2\alpha}t} + C_4 e^{-\frac{1}{2}\lambda t - \frac{1}{2}\sqrt{21\lambda^2+2\alpha}t}, \end{aligned} \tag{IV.7}$$

where

$$C_3 = C_1\frac{a_0}{6\lambda}\left(3\lambda + \sqrt{21\lambda^2+2\alpha}\right), \qquad C_4 = C_2\frac{a_0}{6\lambda}\left(3\lambda - \sqrt{21\lambda^2+2\alpha}\right). \tag{IV.8}$$

Depending on the relative values of the parameters $\Lambda_e = 6\lambda^2$ and α the character of the solutions changes. For the parameters λ, α, as shown on the graph on Fig. 1, in the yellow region between the green and red line we have only damping exponentially decreasing solutions for $r(t)$, while in the grey region below the red line the exponentially vanishing solution is modified by oscillations. On the red line, however, the above form of solutions degenerates, and the correct ones are

$$r(t) = Ce^{-\frac{1}{2}\lambda t}, \qquad r(t) = Cte^{-\frac{1}{2}\lambda t}. \tag{IV.9}$$

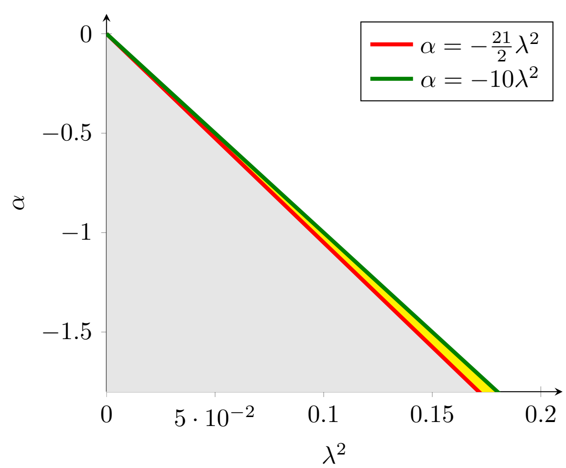

Figure 1: Plot representing sectors in parameters (λ^2, α) with different behaviour of solutions.

On the other hand, we see that the perturbative solutions cannot be extended to $-\infty$ as, independently of the value of the parameters, they then become much bigger than the de Sitter solution. This puts the limits of applicability of the perturbative expansion which is entirely consistent with the dark-energy dominated universe solutions. As a last remark we note that even independently of the value of α perturbations, which are decaying exponentially, are possible for certain values of initial parameters. For example, if at $t = 0$ we set,

$$r(0) = a_0 \left(1 - \frac{\sqrt{21\lambda^2 + 2\alpha}}{3\lambda}\right) s(0),$$

then $C_1 = C_3 = 0$ and the perturbations will be exponentially damped for all range of parameters.

2. *Geometries with positive and negative curvatures $k = \pm 1$*

We start with the easier case of negative curvature, for which the solution of (IV.2) is:

$$a(t) = \frac{1}{\lambda} \sinh\left(\lambda(t - t_0)\right), \tag{IV.10}$$

and in what follows we choose $t_0 = 0$ to simplify the notation.

It is convenient to change the variables and write the equations (IV.4) in $\tau = \sinh(\lambda t)$. Then we obtain,

$$\begin{aligned} \lambda \dot{r}(\tau) &= \left(1 + \frac{\alpha\tau^2}{6\lambda^2(1+\tau^2)}\right) s(\tau) + \frac{\lambda\tau}{1+\tau^2} r(\tau), \\ \lambda \dot{s}(\tau) &= 3\frac{\lambda^2}{\tau^2} r(\tau) - \frac{3\lambda}{\tau} s(\tau). \end{aligned} \tag{IV.11}$$

The above set of equations can be solved explicitly, and the solution for $s(\tau)$ is given by,

$$s(t) = c_1\, {}_2F_1\left(\frac{3}{4} - \zeta, \frac{3}{4} + \zeta; 3; -\tau^2\right) + c_2\, G^{2,0}_{2,2}\left(-\tau^2 \middle| \begin{array}{c} \frac{1}{4} - \zeta, \frac{1}{4} + \zeta \\ -2, 0 \end{array}\right) \tag{IV.12}$$

where ${}_2F_1$ is the hypergeometric function, $G^{2,0}_{2,2}$ is the generalized Meijer's function [23] and

$$\zeta = \frac{\sqrt{21\lambda^2 + 2\alpha}}{4\lambda}.$$

Since the solution is of the Big-Bang cosmology type we shall look for the small t (small τ) behaviour of solutions. Both functions are defined in the region $\tau^2 < 1$ and can be extended analytically to the other values of τ^2, yet $\tau^2 = 1$ is the point at which they are discontinuous or singular. Additionally the Meijer's function has a pole at 0 of order at least 2 unless the parameter ζ is quantized,

$$\zeta = \frac{9}{4} + n, \quad n \in \mathbb{N}, \tag{IV.13}$$

when it becomes regular (though non-zero). For above values of the parameter ζ, the first part of the solution can be rewritten as

$$c_1\, (1 + \tau^2)^{\frac{3}{2}}\, {}_2F_1\left(\frac{9}{2} + n, -n; 3; -\tau^2\right),$$

and the last component is, in fact, a polynomial of degree n.

The possibility of having both solutions regular at $\tau = 0$ means that there exists a nonzero perturbation of the standard solution, which has both perturbations vanishing at the initial time $s(0) = r(0) = 0$. However, the fact that $\tau = 1$ is a singular point of the Meijer's function restricts the possibility of extending the assumed linearised perturbation beyond certain time frame. The long-time behaviour of the solutions that are arbitrary (not necessarily vanishing) at $t = 0$ is similar to the flat case and governed by value of ζ, with asymptotically vanishing solutions for the same range of parameters α, λ as in the $k = 0$ situation.

Finally, for the positive curvature, $k = 1$, the pure dark energy solution is,

$$a(t) = \frac{1}{\lambda}\cosh\left(\lambda(t - t_0)\right), \tag{IV.14}$$

and the small perturbations at $t_0 = 0$ are, again changing the variable to $\tau = \sinh(\lambda t)$,

$$\begin{aligned} \lambda\dot{r}(\tau) &= \frac{1}{\sqrt{1+\tau^2}}\left(\tau + \frac{\alpha(1+\tau^2)}{6\lambda^2\tau}\right)s(\tau) + \frac{\lambda}{\tau}r(\tau), \\ \lambda\dot{s}(\tau) &= \frac{3}{\sqrt{1+\tau^2}}\frac{\lambda^2\tau}{1+\tau^2}r(\tau) - \frac{3\lambda\tau}{1+\tau^2}s(\tau). \end{aligned} \tag{IV.15}$$

which, similarly as in the previous situation, has the solutions that are expressed in terms of the hypergeometric function ${}_2F_1$:

$$s(t) = c_1\, {}_2F_1\left(\frac{3}{4} - \zeta, \frac{3}{4} + \zeta; -\frac{1}{2}; -\tau^2\right) + c_2\, t^3\, {}_2F_1\left(\frac{9}{4} - \zeta, \frac{9}{4} + \zeta; \frac{5}{2}; -\tau^2\right). \tag{IV.16}$$

From the fact that in this case

$$r(\tau) \sim -\frac{\alpha c_1}{6\lambda^3} + \frac{c_2\tau}{\lambda} + O(\tau^2),$$

we deduce that if we require $r(0) = 0$ then $c_1 = 0$. One can easily check that then also $s(0) = 0$, however, both solutions will grow with t. On the other hand, the exponentially decreasing solution requires $c_2 = 0$.

B. Matter dominated universe

In a completely similar manner we consider the limit in a matter-dominated universe, in which we put $\Lambda_e = 0$ and $w = 0$, while $\eta \neq 0$. We start with the Einstein-de Sitter universe, $k = 0$. In this case the standard solution,

$$a(t) = \left(\frac{3}{4}\eta\right)^{\frac{1}{3}} t^{\frac{2}{3}},$$

gives the following equations for $r(t)$ and $s(t)$,

$$\begin{aligned} \dot{r}(t) &= -\frac{1}{2}\frac{\dot{a}(t)}{a(t)}r(t) + \left(\frac{\alpha}{6}\frac{a(t)^2}{\dot{a}(t)} + \dot{a}(t)\right)s(t), \\ \dot{s}(t) &= 3\frac{\dot{a}(t)}{a(t)^2}r(t) - 3\frac{\dot{a}(t)}{a(t)}s(t). \end{aligned} \tag{IV.17}$$

The general solution for $s(t)$ can be expressed in terms of Bessel functions,

$$s(t) = c_1 t^{-\frac{3}{2}} J_{\sqrt{\frac{19}{12}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right) + c_2 t^{-\frac{3}{2}} Y_{\sqrt{\frac{19}{12}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right), \tag{IV.18}$$

and the solution for $r(t)$ can be, consequently derived from the second of (IV.17). In case of negative α the long-time solutions have oscillatory character with the following asymptotic behaviour of their amplitudes:

$$s(t) \sim t^{-2}, \qquad r(t) \sim t^{-\frac{1}{3}},$$

so for $\alpha < 0$ the perturbations decay in t independently of the initial values of the perturbation at any fixed time. Although the matter-dominated universe describes rather later periods in the evolution of the universe, still there exists a solution, which is regular at $t = 0$.

For positive values of α only the second solution, which is exponentially decaying, is an acceptable one as a perturbation, which signifies that for this range of the parameter only specific perturbations are stable.

C. Radiation dominated universe

For this situation (again $\Lambda_e = 0, k = 0$) the standard solution of the Einstein equations is,

$$a(t) = \left(\frac{4}{3}\eta\right)^{\frac{1}{4}} t^{\frac{1}{2}},$$

which gives us the following equations for the perturbations:

$$\begin{aligned} \dot{r}(t) &= -\frac{\dot{a}(t)}{a(t)} r(t) + \left(\frac{\alpha}{6}\frac{a(t)^2}{\dot{a}(t)} + \dot{a}(t)\right) s(t), \\ \dot{s}(t) &= 3\frac{\dot{a}(t)}{a(t)^2} r(t) - 3\frac{\dot{a}(t)}{a(t)} s(t). \end{aligned} \tag{IV.19}$$

The solutions for $s(t)$ is,

$$s(t) = c_1 t^{-\frac{5}{4}} J_{\sqrt{\frac{13}{16}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right) + c_2 t^{-\frac{5}{4}} Y_{\sqrt{\frac{13}{16}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right), \tag{IV.20}$$

with the exact expression for $r(t)$ that can be obtained directly from the second equation.

Again, in the case of $\alpha < 0$ the long-time behaviour of the amplitude of oscillations is

$$s(t) \sim t^{-\frac{7}{4}}, \qquad r(t) \sim t^{-\frac{1}{4}}.$$

However, a very interesting situation occurs near the Big Bang, $t = 0$, as in the best case the solution for $s(t)$ diverges and behaves like $t^{\frac{\sqrt{13}-5}{4}}$, whereas the scale factor $r(t)$ behaves like $t^{\frac{\sqrt{13}-3}{4}}$ and is regular. The same result will be valid for $k = \pm 1$, as the near Big Bang asymptotics of the radiation dominated universe has the same structure.

The explicit solutions for the $k = -1$ geometry are in terms of the confluent Heun functions and the long-time dependence of the perturbations will be again similar for $\alpha < 0$ as is suggested by a brief numerical analysis of example solutions.

As the solutions for $k = 1$ are cyclic, the long-term asymptotic of the perturbations does not make sense in this case.

V. SUMMARY AND OUTLOOK

The simplest almost-commutative geometry of the two-sheeted universe, which is motivated by the Connes-Lott idea [11] is an interesting model to study its potential relevance not only for the particle physics but also for its implication to the large-scale structure of the Universe. We have shown that an abstract model, with a more general type of metric structure that is not necessarily a product structure allows a two-metric theory, which is very similar to the bimetric theory of gravity. Although we are aware that both the interaction structure as well as the interpretation of the model's origin are quite different there are striking similarities in the potential term of the action. It shall be noted that models originating from quantum deformations of spacetime have a similar feature of two metrics although their origins are different [24].

Leaving the full model that was developed for the particle interactions [8, 9] aside and concentrating first on a simplified one, we have focused on a primary question of stability of classical Friedmann-Lemaître-Robertson-Walker solutions. To be more precise, our idea was to check whether for some range of parameters a small perturbation in the Dirac operators making the full one, and hence the metrics different from each other on the two sheets of the Universe, will diverge or collapse.

Our conclusion is that for the considered range of models, including flat and curved spatial geometries with dark-energy, radiation or matter dominance there exist a range of parameters so that the symmetric solution (product geometry) is dynamically stable. Our analysis confirms but hugely extends the earlier indications [12] by allowing both the scale factors as well as lapse functions to vary. The stability of the cosmological solutions suggests that the models with two metrics are admissible from the physical point of view and are an interesting modification of geometry that may be used in future models.

This has an important bearing on the physical consequences of the model. First of all, cosmological observables like redshift and observable Hubble constant will be related to the background standard Friedmann-Lemaître-Robertson-Walker solution. This follows from the fact that both light and matter will couple (as argued in section III) to both metrics and, taking into account that in most models the difference between metrics is decreasing as the Universe evolves, only the average (background) scale factor $a(t)$ will determine the observable redshift. However, one can speculate that a possible sign of the fluctuating two metrics might be seen in physical effects that couple only to one metric (as might be the case of massless Majorana particles) or couple to metrics in a nonlinear way.

The constructed (simplified) model is predominantly based on the idea that allowed to explain the appearance of Higgs field and Higgs quartic symmetric-breaking potential from purely geometric considerations as a form of generalized gauge theory. Transferring this concept to the theory of metric and generalized general relativity appears to be a natural an well-motivated physical step. Unlike in the bimetric theory, here the interaction terms between the two metrics are completely determined by the structure of the theory yet are not computable in full generality. This prevents us from an analysis of the possibility of ghost-free sectors in the way it was done for bimetric theories [7, 25] . Nevertheless, since the model has strong features similar to bimetric gravity (as we have stressed in II G), in particular, even though the effective interaction potential between the two metric is not a symmetric polynomial of $\sqrt{g_1^{-1}g_2}$ but rather a rational function, where the nominator and denominator are of this form, we expect that a similar result will hold.

Apart from the fundamental questions of physical consistency and interpretation of the degrees of freedom of the theory there are still several questions that remain open. First of all, in case of small deviations from the product geometry it is interesting whether they might have some observable physical consequences both in the pure gravity sector as well as in the sector of the matter and radiation. Though this might be considered as pure speculation, such fluctuations of the metrics, if existing in the radiation era, might be linked to some parity anisotropies [26] in the Cosmic Microwave Background radiation. Another possible sector of the theory to explore are solutions with singularities like black holes. All such ideas need to be explored carefully in future studies.

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Appendix A: SYMBOLS OF THE OPERATOR D^{-2}

Suppose P and Q are two pseudodifferential operators with symbols

$$\sigma_P(x,\xi) = \sum_\alpha \sigma_{P,\alpha}(x)\xi^\alpha, \qquad \sigma_Q(x,\xi) = \sum_\beta \sigma_{Q,\beta}(x)\xi^\beta, \tag{A.1}$$

respectively, where α, β are multiindices. The composition rule takes the following form [27]

$$\sigma_{PQ}(x,\xi) = \sum_\gamma \frac{(-i)^{|\gamma|}}{\gamma!}\partial^\xi_\gamma \sigma_P(x,\xi)\partial_\gamma \sigma_Q(x,\xi), \tag{A.2}$$

where ∂^ξ_a denotes partial derivative with respect to coordinate of the cotangent bundle.

Let us consider the case when $P = D^{-2}$ and $Q = D^2$. Since D^2 has a symbol

$$\sigma_{D^2}(x,\xi) = \mathfrak{a}_2 + \mathfrak{a}_1 + \mathfrak{a}_0, \tag{A.3}$$

then D^{-2} has to have a symbol of the form

$$\sigma_{D^{-2}}(x,\xi) = \mathfrak{b}_0 + \mathfrak{b}_1 + \mathfrak{b}_2 + ..., \tag{A.4}$$

where $\mathfrak{b}_k$ is homogeneous of order $-2-k$.

Inserting these expressions into (A.2) and taking homogeneous parts of order $0, -1$ and -2 we get the following set of equations:

$$\begin{aligned}
&\mathfrak{b}_0\mathfrak{a}_2 = 1, \\
&\mathfrak{b}_0\mathfrak{a}_1 + \mathfrak{b}_1\mathfrak{a}_2 - i\partial^\xi_a(\mathfrak{b}_0)\partial_a(\mathfrak{a}_2) = 0, \\
&\mathfrak{b}_2\mathfrak{a}_2 + \mathfrak{b}_1\mathfrak{a}_1 + \mathfrak{b}_0\mathfrak{a}_0 - i\partial^\xi_a(\mathfrak{b}_0)\partial_a(\mathfrak{a}_1) - i\partial^\xi_a(\mathfrak{b}_1)\partial_a(\mathfrak{a}_2) - \frac{1}{2}\partial^\xi_a\partial^\xi_b(\mathfrak{b}_0)\partial_a\partial_b(\mathfrak{a}_2) = 0,
\end{aligned} \tag{A.5}$$

From these relations we get

$$\begin{aligned}
\mathfrak{b}_0 &= \mathfrak{a}_2^{-1},\\
\mathfrak{b}_1 &= -\left(\mathfrak{b}_0\mathfrak{a}_1 - i\partial_a^{\xi}(\mathfrak{b}_0)\partial_a(\mathfrak{a}_2)\right)\mathfrak{b}_0,\\
\mathfrak{b}_2 &= -\left(\mathfrak{b}_1\mathfrak{a}_1 + \mathfrak{b}_0\mathfrak{a}_0 - i\partial_a^{\xi}(\mathfrak{b}_0)\partial_a(\mathfrak{a}_1) - i\partial_a^{\xi}(\mathfrak{b}_1)\partial_a(\mathfrak{a}_2) - \frac{1}{2}\partial_a^{\xi}\partial_b^{\xi}(\mathfrak{b}_0)\partial_a\partial_b(\mathfrak{a}_2)\right)\mathfrak{b}_0.
\end{aligned} \tag{A.6}$$

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2.2.2 Analysis of the interaction term

The action for the doubled geometry model, discussed in the previous subsection, contains a potential term that possesses several features of the original bimetric theory, but its exact analytical form is different than the one present in bimetric models. However, one can ask if these two approaches coincide in certain situations. To answer this question we consider the doubled geometry model with arbitrary metrics, i.e. not necessarily being of the FLRW type. Our first goal is to examine this model in a situation when both the metrics differ infinitesimally from the Euclidean ones. In other words, we assume that each of the metrics is of the form $g_{ij} = \delta_{ij} + \epsilon h_{ij}$. Since we are interested in pure interactions only, we can assume that the functions h are just constants. We compute the spectral action for such a choice of geometries and then expand it in the perturbation parameter up to the fourth order.

The next step is to similarly expand also the action for the bimetric model and try to match the coefficients by comparing terms of the expansions for these two models. We conclude that already in the second order in the perturbation parameter these two are different. The doubled geometry model is therefore distinct from the bimetric gravity, however, it possesses most of its features. On the other hand, the original bimetric model was constructed without a solid geometric motivation, and the form of action was rather postulated than derived. Therefore, it seems reasonable to expect that our model, which shares the main characteristics with the bimetric model, might be interesting from the cosmological perspective.

We propose a possible physical realization of such geometry. One can think of two four-dimensional branes living in a higher dimensional space. The interaction term, which is actually of the Higgs type (cf. the discussion of the Standard Model in the previous sections), may be thought of as the one describing a certain type of interactions between these two parallel universes - the two copies of the spacetime.

Independently of its interpretation, it is crucial to have a compact representation of the action for the perturbed model s.t. each of its terms is an invariant of the matrix $\sqrt{g_2^{-1}g_1}$. We find this presentation, which could be used in the future for further analysis of our model from the point of view of possible cosmological applications.

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Spectral interaction between universes

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Abstract

We derive a perturbative formula for the direct interaction between two four-dimensional geometries. Based on the spectral action principle we give an explicit potential up to the third order perturbation around the flat vacua. We present the leading terms of the interaction as polynomials of the invariants of the two metrics and compare the expansion to the models of bimetric gravity.

1 Introduction

One of the most significant achievements of modern physics is geometry's spectacular success in describing the large-scale structure and the evolution of the Universe thanks to general relativity. On the subatomic scale, the geometric picture of gauge theories establishes the natural framework for fundamental particle interactions. Although the common unifying scheme for both, apparently different, types of interactions is not yet known, there exist various approaches that aim to bridge the gap. Noncommutative geometry, which changes the way of approach by making the differential operators as fundamental objects can, at least on the classical level, treat the gauge fields as well as the metric as different fields that parametrize the real physical object, the Dirac operator [1, 2, 3].

The theory, when applied to the Standard Model of particle interactions can explain, in a purely geometric way, the existence of the Higgs field and the appearance of symmetry-breaking potential. However, the necessary element, that has to be added, includes a geometry of discrete-type, which is described as a finite-dimensional matrix algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. In a simplified model (which ignores the strong interactions and treats the weak interactions as electromagnetic) one can reduce this algebra to $\mathbb{C} \oplus \mathbb{C}$ leading to a simple geometry of a product of the four-dimensional spin manifold with two points [4, 5].

The model, which looks like a two-sheeted geometry and can be compared to the model with two four-dimensional branes or the boundary of a thin domain wall of five dimensions in the bulk (see [6] for vast literature on the latter topic). This extends the image of the universe as a brane in the bulk with the possibility of the system of a pair of interacting branes. Since the interactions with the bulk and between the branes influence the physics it is natural to ask what is the origin of the interactions and whether it is possible to have it of purely geometric origin. The answer comes again through the tools noncommutative geometry. We assume that the interaction between the fields on the two sheets is mediated by the Higgs field, which itself is related to the metric and the connection on the discrete component of the geometry. Following this idea, we can, in principle, derive an explicit and unambiguous interaction between the geometries alone, depending only on the metrics.

In the constructions so far, one usually assumed the natural product-type geometry (product-type Dirac operators), which, after applying the spectral action procedure [7, 8] led to the standard Einstein-Hilbert action for the metric, identical on the two universes. Yet this is not the most general form of the Dirac operator and different metrics on the two separate universes are admissible [9]. Together with the Higgs-type field that mediates between the two geometries one can obtain an interaction term between the two metrics, leading to an interesting class of models, which appear to be viable from the point of view of cosmological models [10].

The general type of the interaction term for two arbitrary metrics is not explicitly computable even in the simple setup. An exact answer was obtained only for the Friedmann-Lemaître-Robertson-Walker (FRLW) type of Euclidean metrics [10] which allowed to study the stability of solutions of cosmological evolution equations. Interestingly, the obtained models resemble the so-called bimetric gravity [11], which is a good candidate to potentially solve the puzzle of dark matter in accordance with the cosmological data [12, 13, 14] and does not suffer from Boulware-Deser ghost problem [11, 15]. However, the bimetric gravity lacks a geometrical interpretation and the second metric-like field is not well justified from the standpoint of Riemannian geometry.

As the noncommutative geometry motivated model of two-sheeted space with two metrics yields a similar theory, with a full diffeomorphism invariance, it is natural to ask how do these models differ. In particular, in contrast to bimetric theory, the two-sheeted geometry spectral action principle fixes uniquely (up to multiplicative constant) the interaction term between the metrics. Our previous analysis of the FLRW type geometries allowed us to give only a partial answer about the similarities and differences of the two approaches.

In this note, we derive an explicit form of the spectral action for the infinitesimal perturbation of the flat metric (in the Euclidean setup) on the two-sheeted geometry (up to the fourth-order) and compare it with the general action proposed for bimetric gravity up to the first three orders. This demonstrates that a simple model of noncommutative geometry allows direct and generic interaction between universes (branes) and opens a possibility to study the general properties of such models.

2 The interaction of geometries - a general construction

Noncommutative geometry allows us to generalize classical concepts from differential geometry in a systematic way. The fundamental object is a spectral triple, which is a system (A, H, D) consisting of a unital $*$-algebra represented (faithfully) on a Hilbert space H and the Dirac operator D, which is essentially self-adjoint on H. Several compatibility conditions are assumed, such as the compactness of the Dirac operator's resolvent, the boundedness of certain commutators, and so on. The spectral triple $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, g), D)$ encoding the geometry of the (compact, spin) Riemannian manifold $(\mathcal{M}, g)$ is the canonical example. Locally, $D = i\gamma^\mu(\partial_\mu + \omega_\mu)$, where ω_μ is the spin connection on the spinor bundle over $(\mathcal{M}, g)$. Another example is almost-commutative geometry, which is defined as the product of the canonical spectral triple and some finite one, (A_F, H_F, D_F), with A_F and H_F being finite dimensional, and D_F being a (matrix) operator acting on H_F. The corresponding Dirac operator has the form $D \otimes 1 + \gamma_M \otimes D_F$ with γ_M being the canonical grading on $\mathcal{M}$ (that is, at a given point, $\gamma_M = \gamma_5$ in the associated Clifford algebra). Spectral triples of this product type of geometry were successfully applied to the description of the Standard Model of particle physics [16] and provided a geometric understanding of this model.

The natural generalization of the almost-commutative product-like geometry for the Riemannian (four-dimensional) manifold $\mathcal{M}$ and the finite space $\mathbb{Z}_2$ is the one with the Dirac operator not being of the product type. This defines the so-called *doubled geometry* [9, 10]. More precisely, for this spectral triple the Dirac operator is taken to be of the form

$$\mathcal{D} = \begin{pmatrix} D_1 & \gamma\Phi \\ \gamma\Phi^* & D_2 \end{pmatrix}, \tag{1}$$

where D_1 and D_2 are the two usual Dirac operators for the two copies of the (spin) Riemannian manifold $\mathcal{M}$, with metrics g_1 and g_2, respectively. Here γ is an operator that squares to $\kappa = \pm 1$ and is a generalization of the usual grading on the canonical spectral triple (see [10] for detailed discussion of the role and origin of this operator). We assume that γ is Hermitian and anticommutes with all the γ^a matrices which are taken to be anti-Hermitian and satisfy $\gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab}$.

For a given metric g on the manifold $\mathcal{M}$, the Dirac operator can be written explicitely as

$$D = \gamma^a dx^\mu(\theta^a)\frac{\partial}{\partial x^\mu} + \frac{1}{4}\gamma^c\omega_{cab}\gamma^a\gamma^b, \tag{2}$$

where $\{\theta^a\}$ is the orthogonal coframe, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \theta^a\theta^a$, and the (coefficients of the) spin connection can be computed by using the relation $d\theta^a = \omega^{ab}\wedge\theta^b$.

3 The perturbative interaction of two metric geometries.

We assume that our manifold is a four-dimensional Euclidean torus [1] with the natural choice of global coordinates and with the metric that is an infinitesimal perturbation of the flat one,

$$g_{ij} = \delta_{ij} + \epsilon h_{ij}, \tag{3}$$

with some h_{ij} and the perturbation parameter ϵ. The inverse metric, g^{ij}, is, up to ϵ^4:

$$g^{ij} = \delta^{ij} - \epsilon h^{ij} + \epsilon^2 h^{ik}h_k^{\,j} - \epsilon^3 h^{ik}h_{km}h^{mj} + \epsilon^4 h^i{}_j h^{jl}h_{lm}h^{mk}. \tag{4}$$

In what follows we will be interested in the form of the potential term, therefore we are allowed to put $h_{ij} = \mathrm{const}$ as no derivatives of the metric enter. Therefore, the Dirac operator D, again up to ϵ^4, becomes:

$$D = \gamma^i\left(\delta^j_i - \frac{1}{2}\epsilon h^j{}_i + \frac{3}{8}\epsilon^2 h^j{}_k h^k{}_i - \frac{5}{16}\epsilon^3 h^j{}_k h^k{}_l h^l{}_i + \frac{35}{128}\epsilon^4 h^j{}_l h^l{}_k h^k{}_n h^n{}_i\right)\partial_j. \tag{5}$$

Since we are working with the constant metric we take as the Hilbert space the two copies of the square-summable sections of the usual spinor bundle over the four-torus with respect to the flat metric, $\mathcal{H} = L^2(S)\otimes\mathbb{C}^2$. This facilitates the computations and does not single out any of the geometries as a preferred one.

For the two-sheeted metric geometry with the Higgs-type field, as described in Sec. 2, the Dirac-type operator in noncommutative geometry has the form,

$$\begin{aligned}\mathcal{D} &= \gamma^j\partial_j + \gamma F - \frac{1}{2}\epsilon\gamma^i H^j_i\partial_j + \frac{3}{8}\epsilon^2\gamma^i H^{jk}H_{ki}\partial_j \\ &\quad - \frac{5}{16}\epsilon^3\gamma^i H^j_k H^k_l H^l{}_i\partial_j + \frac{35}{128}\epsilon^4\gamma^i H^j_k H^k_l H^l{}_n H^n_i\partial_j,\end{aligned} \tag{6}$$

where

$$H_{jk} = \begin{pmatrix} h_{1jk} & \\ & h_{2jk}\end{pmatrix}, \qquad F = \begin{pmatrix} & \Phi \\ \Phi^* & \end{pmatrix}, \tag{7}$$

and we can also assume that the field Φ is constant (since we do not investigate the dynamical terms). As a result,

$$\begin{aligned}\mathcal{D}^2 = &-\partial_j^2 + \kappa F^2 + \epsilon H^{jk}\partial_j\partial_k - \frac{\epsilon}{2}\gamma\gamma^j[F, H^k_j]\partial_k \\ &- \epsilon^2 H^j_l H^{lk}\partial_j\partial_k + \frac{3}{8}\epsilon^2\gamma\gamma^j[F, H^{kl}H_{lj}]\partial_k \\ &+ \epsilon^3 H^j_n H^n_l H^{lk}\partial_j\partial_k - \frac{5}{16}\epsilon^3\gamma\gamma^j[F, H^k_l H^l{}_n H^n_j]\partial_k \\ &- \epsilon^4 H^j_m H^m_l H^l{}_n H^{nk}\partial_j\partial_k + \frac{35}{128}\epsilon^4\gamma\gamma^j[F, H^k_l H^l{}_m H^m_n H^n_j]\partial_k,\end{aligned} \tag{8}$$

[1] As all of the obtained terms are local, this assumption is only technical and the results will hold for any compact manifold.

where $\kappa = \pm 1$ depending on the properties of the grading γ.

The computation of the interaction term follows the general principle of the spectral action. Technically, to obtain the Einstein-Hilbert action and its generalization we compute the Wodzicki residue of the inverse of $\mathcal{D}^2$ [17]. The procedure uses the explicit computation of the symbols of the pseudodifferential operator $\mathcal{D}^{-2}$ and the integration over the cosphere.

The homogeneous parts of the symbol of the differential operator $\mathcal{D}^2$ are,

$$\begin{aligned}
\mathfrak{a}_2 =& \|\xi\|^2 - \epsilon H^{jk}\xi_j\xi_k + \epsilon^2 H^j_{\ l} H^{lk}\xi_j\xi_k - \epsilon^3 H^j_{\ n} H^n_{\ l} H^{lk}\xi_j\xi_k + \epsilon^4 H^j_{\ m} H^m_{\ l} H^l_{\ n} H^{nk}\xi_j\xi_k,\\
\mathfrak{a}_1 =& -\frac{i}{2}\epsilon\gamma\gamma^j[F, H^k_{\ j}]\xi_k + \frac{3i}{8}\epsilon^2\gamma\gamma^j[F, H^{kl}H_{lj}]\xi_k - \frac{5i}{16}\epsilon^3\gamma\gamma^j[F, H^k_{\ l}H^l_{\ n}H^n_{\ j}]\xi_k\\
&+ \frac{35i}{128}\epsilon^4\gamma\gamma^j[F, H^k_{\ l}H^l_{\ m}H^m_{\ n}H^n_{\ j}]\xi_k,\\
\mathfrak{a}_0 =& \kappa F^2.
\end{aligned} \tag{9}$$

The symbols of its inverse, $\sigma_{\mathcal{D}^{-2}} = \mathfrak{b}_{-2} + \mathfrak{b}_{-3} + \mathfrak{b}_{-4} + \ldots$, are much more complicated, with the principal symbol,

$$\begin{aligned}
\mathfrak{b}_{-2} = & \frac{1}{\|\xi\|^2}\left(1 + \epsilon H^{jk}\frac{\xi_j\xi_k}{\|\xi\|^2} + \epsilon^2\left(H^{jk}H^{mn}\frac{\xi_j\xi_k\xi_m\xi_n}{\|\xi\|^4} - H^j_{\ l}H^{lk}\frac{\xi_j\xi_k}{\|\xi\|^2}\right)\right.\\
&+ \epsilon^3\left(H^{jk}H^{mn}H^{rs}\frac{\xi_j\xi_k\xi_m\xi_n\xi_r\xi_s}{\|\xi\|^6} - 2H^{jk}H^m_{\ l}H^{ln}\frac{\xi_j\xi_k\xi_m\xi_n}{\|\xi\|^4} + H^j_{\ n}H^n_{\ l}H^{lk}\frac{\xi_j\xi_k}{\|\xi\|^2}\right)\\
&+\epsilon^4\left(H^{jk}H^{mn}H^{rs}H^{pq}\frac{\xi_j\xi_k\xi_m\xi_n\xi_r\xi_s\xi_p\xi_q}{\|\xi\|^8} - 3H^j_{\ l}H^{lk}H^{mn}H^{rs}\frac{\xi_j\xi_k\xi_m\xi_n\xi_r\xi_s}{\|\xi\|^6}\right.\\
&\left.\left.+ \left(2H^j_{\ n}H^n_{\ l}H^{lk}H^{rs} + H^j_{\ l}H^{lk}H^r_{\ n}H^{ns}\right)\frac{\xi_j\xi_k\xi_r\xi_s}{\|\xi\|^4} - H^j_{\ m}H^m_{\ l}H^l_{\ n}H^{nk}\frac{\xi_j\xi_k}{\|\xi\|^2}\right)\right).
\end{aligned} \tag{10}$$

For the $\mathfrak{b}_{-4}$ we are, effectively, interested only in its component $\mathfrak{b}'_{-4}$ that contains the interaction terms between the metrics (there will be terms that are proportional to the volume and the separate Einstein-Hilbert terms for each metric) that arises (for the constant metrics) exclusively from the product $\mathfrak{b}_{-2}\mathfrak{a}_1\mathfrak{b}_{-2}\mathfrak{a}_1\mathfrak{b}_{-2}$ term. To obtain the final expression we already use the properties of the trace over the algebra of 2×2 matrices as well as over the Clifford algebra, which significantly simplifies the number of terms.

For this part, we obtain,

$$\begin{aligned}
&\mathrm{Tr}_{Cl}\mathrm{Tr}(\mathfrak{b}'_{-4}) = -\frac{\kappa}{\|\xi\|^6}\Bigg\{\epsilon^2\,\mathrm{Tr}\left([F, H^m_{\ j}][F, H^{nj}]\right)\xi_m\xi_n\\
&+\epsilon^3\,\mathrm{Tr}\left(3[F, H^k_{\ j}][F, H^{nj}]H^{st}\frac{\xi_k\xi_n\xi_s\xi_t}{\|\xi\|^2} - \frac{3}{2}[F, H^k_{\ j}][F, H^n_{\ m}H^{mj}]\xi_n\xi_k\right)\\
&+\epsilon^4\,\mathrm{Tr}\Bigg[\left(4[F, H^k_{\ j}][F, H^{pj}]H^{rs}H^{mn} + 2[F, H^k\ j]H^{rs}[F, H^{pj}]H^{mn}\right)\frac{\xi_k\xi_p\xi_r\xi_s\xi_m\xi_n}{\|\xi\|^4}\\
&\quad - 3[F, H^k\ j][F, H^{pj}]H^r_{\ n}H^{ns}\frac{\xi_k\xi_p\xi_r\xi_s}{\|\xi\|^2}\\
&\quad - \frac{9}{4}\left([F, H^k_{\ j}][F, H^{nm}H^{\ j}_{m}]H^{rs} + [F, H^{nm}H^{\ j}_{m}][F, H^k_{\ j}]H^{rs}\right)\frac{\xi_k\xi_n\xi_r\xi_s}{\|\xi\|^2}\\
&\quad + \frac{5}{4}[F, H^k_{\ j}][F, H^n_{\ m}H^m_{\ r}H^{rj}]\xi_k\xi_n + \frac{9}{16}[F, H^{kl}H_{lj}][F, H^{np}H_{pj}]\xi_k\xi_n\Bigg]\Bigg\}.
\end{aligned} \tag{11}$$

After integrating the result over the cosphere (the integrals we have used are in the appendix A) and the manifold, and further using the symmetry of the perturbation terms, the interaction term reduces to,

$$
\begin{aligned}
S(h_1,h_2) \sim & \epsilon^2\,\mathrm{Tr}(h_2-h_1)^2 \\
& +\epsilon^3\left[\frac{1}{4}\mathrm{Tr}(h_2-h_1)^2\mathrm{Tr}(h_1+h_2)-\mathrm{Tr}\left[(h_2-h_1)^2(h_1+h_2)\right]\right] \\
& +\epsilon^4\left\{\frac{1}{24}\mathrm{Tr}(h_2-h_1)^2\left[(\mathrm{Tr}\,h_1)^2+(\mathrm{Tr}\,h_2)^2+(\mathrm{Tr}\,h_1)(\mathrm{Tr}\,h_2)\right.\right. \\
& \qquad \left. -4\mathrm{Tr}(h_1^2+h_2^2)+2\mathrm{Tr}(h_1h_2)\right] \\
& -\frac{1}{6}\mathrm{Tr}(h_2-h_1)^4+\frac{5}{4}\mathrm{Tr}\left[(h_2-h_1)(h_2^3-h_1^3)\right]-\frac{3}{16}\mathrm{Tr}(h_2^2-h_1^2)^2 \\
& +\frac{1}{12}\left[(\mathrm{Tr}\,h_1)\mathrm{Tr}[(h_2-h_1)^2h_1]+(\mathrm{Tr}\,h_2)\mathrm{Tr}[(h_2-h_1)^2h_2]\right] \\
& \left.-\frac{7}{24}\mathrm{Tr}(h_1+h_2)\mathrm{Tr}[(h_2-h_1)^2(h_2+h_1)]\right\}.
\end{aligned}
\tag{12}
$$

This expression has a much simpler form when replacing the h_1, h_2 perturbations by their linear combinations. With

$$W_- = h_2 - h_1, \qquad W_+ = h_2 + h_1,$$

we have:

$$
\begin{aligned}
S(h_1,h_2) \sim & \;\epsilon^2\,\mathrm{Tr}(W_-)^2 \\
& +\epsilon^3\,\frac{1}{4}\left(\mathrm{Tr}(W_-)^2\mathrm{Tr}(W_+)-4\mathrm{Tr}(W_+W_-^2)\right) \\
& +\epsilon^4\,\left(\frac{1}{32}\mathrm{Tr}(W_-^2)\,(\mathrm{Tr}\,W_+)^2+\frac{1}{96}\mathrm{Tr}(W_-^2)\,(\mathrm{Tr}\,W_-)^2-\frac{1}{16}\mathrm{Tr}(W_-^2)\mathrm{Tr}(W_+^2)\right. \\
& -\frac{5}{48}\left(\mathrm{Tr}\,W_-^2\right)^2+\frac{7}{48}\mathrm{Tr}(W_-^4)+\frac{3}{4}\mathrm{Tr}(W_+^2W_-^2) \\
& \left.+\frac{1}{24}\mathrm{Tr}(W_-)\mathrm{Tr}(W_-^3)-\frac{1}{4}\mathrm{Tr}(W_+)\mathrm{Tr}(W_-^2W_+)\right).
\end{aligned}
\tag{13}
$$

4 Comparison with bimetric gravity models

The commonly assumed interaction part between the two metrics g_1, g_2 in the bimetric gravity models [11, 12] is of the form

$$S_{\mathrm{int}} \sim \int d^4x\sqrt{\det g_2}\left(\sum_{n=0}^{4}\beta_n e_n(\mathbb{X})\right), \tag{14}$$

where the matrix $\mathbb{X} = \sqrt{g_2^{-1}g_1}$, and the constants β_n are free parameters of the model. The invariant functions e_n are,

$$
\begin{aligned}
& e_0(\mathbb{X}) = 1, && e_1(\mathbb{X}) = \mathrm{Tr}(\mathbb{X}), \\
& e_2(\mathbb{X}) = \frac{1}{2}\left((\mathrm{Tr}(\mathbb{X}))^2-\mathrm{Tr}(\mathbb{X}^2)\right), && e_4(\mathbb{X}) = \det(\mathbb{X}),
\end{aligned}
$$

$$e_3(\mathbb{X}) = \frac{1}{6}\left((\mathrm{Tr}(\mathbb{X}))^3-3\mathrm{Tr}(\mathbb{X})\mathrm{Tr}(\mathbb{X}^2)+2\mathrm{Tr}(\mathbb{X}^3)\right).$$

Let us expand the above action in ϵ when $g_{1ij} = \delta_{ij} + \epsilon h_{1ij}$ and $g_{2ij} = \delta_{ij} + \epsilon h_{2ij}$. First, we compute,

$$
\begin{aligned}
\sqrt{\det g_2} = & \, 1+\frac{1}{2}\epsilon\mathrm{Tr}(h_2)+\frac{1}{8}\epsilon^2\left((\mathrm{Tr}(h_2))^2-2\mathrm{Tr}(h_2^2)\right) \\
& +\frac{1}{48}\epsilon^3\left[(\mathrm{Tr}(h_2))^3-6\mathrm{Tr}(h_2)\mathrm{Tr}(h_2^2)+8\mathrm{Tr}(h_2^3)\right],
\end{aligned}
\tag{16}
$$

and

$$g_2^{ij}g_{1jk} = \delta^i{}_k + \epsilon\left(h_1 - h_2\right)^i{}_k + \epsilon^2\left(h_2^2 - h_2h_1\right)^i{}_k + \epsilon^3\left(h_2^2h_1 - h_2^3\right)^i{}_k, \tag{17}$$

so that

$$\begin{aligned}\mathbb{X}^i{}_k = \left(\sqrt{g_2^{-1}g_1}\right)^i{}_k &= \delta^i{}_k + \frac{1}{2}\epsilon(h_1 - h_2)^i{}_k + \frac{1}{8}\epsilon^2\left[3h_2^2 - h_1^2 + h_1h_2 - 3h_2h_1\right]^i{}_k \\ &+ \frac{1}{16}\epsilon^3\left[h_1^3 + h_2h_1^2 - h_1^2h_2 - h_2h_1h_2 + h_1h_2h_1 + 5h_2^2h_1 - h_1h_2^2 - 5h_2^3\right]^i{}_k.\end{aligned} \tag{18}$$

Finally, we can expand all traces,

$$\begin{aligned}\mathrm{Tr}(\mathbb{X}) &= 4 + \frac{1}{2}\epsilon\mathrm{Tr}(h_1 - h_2) + \frac{1}{8}\epsilon^2\left[3\mathrm{Tr}(h_2^2) - \mathrm{Tr}(h_1^2) - 2\mathrm{Tr}(h_1h_2)\right] \\ &\quad + \frac{\epsilon^3}{16}\left[\mathrm{Tr}(h_1^3) + \mathrm{Tr}(h_2h_1^2) + 3\mathrm{Tr}(h_2^2h_1) - 5\mathrm{Tr}(h_2^3)\right] = e_1(\mathbb{X}), \\ \mathrm{Tr}(\mathbb{X}^2) &= 4 + \epsilon\mathrm{Tr}(h_1 - h_2) + \epsilon^2\left[\mathrm{Tr}(h_2^2) - \mathrm{Tr}(h_1h_2)\right] + \epsilon^3\left[\mathrm{Tr}(h_2^2h_1) - \mathrm{Tr}(h_2^3)\right], \\ \mathrm{Tr}(\mathbb{X}^3) &= 4 + \frac{3}{2}\epsilon\mathrm{Tr}(h_1 - h_2) + \frac{3}{8}\epsilon^2\left[5\mathrm{Tr}(h_2^2) + \mathrm{Tr}(h_1^2) - 6\mathrm{Tr}(h_1h_2)\right] \\ &\quad + \frac{1}{16}\epsilon^3\left[-\mathrm{Tr}(h_1^3) - 35\mathrm{Tr}(h_2^3) - 9\mathrm{Tr}(h_1^2h_2) + 45\mathrm{Tr}(h_1h_2^2)\right],\end{aligned} \tag{19}$$

and in the end we have the expansion of all invariants e_k:

$$\begin{aligned}e_2(\mathbb{X}) = 6 &+ \frac{3}{2}\epsilon\,\mathrm{Tr}(h_1 - h_2) + \frac{1}{8}\epsilon^2\,\left[(\mathrm{Tr}(h_1))^2 + (\mathrm{Tr}(h_2))^2 - 2\mathrm{Tr}(h_1)\mathrm{Tr}(h_2)\right. \\ &\qquad\left. + 8\mathrm{Tr}(h_2^2) - 4\mathrm{Tr}(h_1^2) - 4\mathrm{Tr}(h_1h_2)\right] \\ &+ \frac{1}{16}\epsilon^3\,\left[4\mathrm{Tr}(h_1^3) + 4\mathrm{Tr}(h_2^2h_1) - 12\mathrm{Tr}(h_2^3) + 4\mathrm{Tr}(h_2h_1^2)\right. \\ &\qquad - \mathrm{Tr}(h_1)\mathrm{Tr}(h_1^2) - 2\mathrm{Tr}(h_1)\mathrm{Tr}(h_1h_2) + 3\mathrm{Tr}(h_1)\mathrm{Tr}(h_2^2) \\ &\qquad\left. + \mathrm{Tr}(h_2)\mathrm{Tr}(h_1^2) + 2\mathrm{Tr}(h_2)\mathrm{Tr}(h_1h_2) - 3\mathrm{Tr}(h_2)\mathrm{Tr}(h_2^2)\right],\end{aligned} \tag{20}$$

$$\begin{aligned}e_3(\mathbb{X}) = 4 &+ \frac{3}{2}\epsilon\,\mathrm{Tr}(h_1 - h_2) + \frac{1}{8}\epsilon^2\,\left[-5\mathrm{Tr}(h_1^2) - 2\mathrm{Tr}(h_1h_2) + 7\mathrm{Tr}(h_2^2)\right. \\ &\qquad\left. + 2(\mathrm{Tr}(h_1))^2 + 2(\mathrm{Tr}(h_2))^2 - 4\mathrm{Tr}(h_2)\mathrm{Tr}(h_1)\right] \\ &+ \frac{1}{48}\epsilon^3\,\left[17\mathrm{Tr}(h_1^3) - 29\mathrm{Tr}(h_2^3) + 9\mathrm{Tr}(h_1^2h_2) + 3\mathrm{Tr}(h_1h_2^2)\right. \\ &\qquad - 9\mathrm{Tr}(h_1)\mathrm{Tr}(h_1^2) - 6\mathrm{Tr}(h_1)\mathrm{Tr}(h_1h_2) + 15\mathrm{Tr}(h_1)\mathrm{Tr}(h_2^2) + (\mathrm{Tr}(h_1))^3 \\ &\qquad + 9\mathrm{Tr}(h_2)\mathrm{Tr}(h_1^2) + 6\mathrm{Tr}(h_2)\mathrm{Tr}(h_1h_2) - 15\mathrm{Tr}(h_2)\mathrm{Tr}(h_2^2) \\ &\qquad\left. - 3\mathrm{Tr}(h_2)(\mathrm{Tr}(h_1))^2 + 3\mathrm{Tr}(h_1)(\mathrm{Tr}(h_2))^2 - (\mathrm{Tr}(h_2))^3\right].\end{aligned} \tag{21}$$

For the last ivariant, arising from the determinant, we have,

$$\begin{aligned}\mathrm{Det}(1 + \epsilon A + \epsilon^2 B + \epsilon^3 C) =& 1 + \epsilon\mathrm{Tr}(A) + \epsilon^2\left(\mathrm{Tr}(B) + \frac{1}{2}\left((\mathrm{Tr}(A))^2 - \mathrm{Tr}(A^2)\right)\right) \\ &+ \epsilon^3\Bigg(\mathrm{Tr}(C) + \mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB) \\ &\quad + \frac{1}{6}(\mathrm{Tr}(A))^3 - \frac{1}{2}\mathrm{Tr}(A^2)\mathrm{Tr}(A) + \frac{1}{3}\mathrm{Tr}(A^3)\Bigg),\end{aligned} \tag{22}$$

leading to,

$$\begin{aligned}e_4(\mathbb{X}) = 1 &+ \frac{1}{2}\epsilon\mathrm{Tr}(h_1 - h_2) + \frac{1}{8}\epsilon^2\left((\mathrm{Tr}(h_1 - h_2))^2 + 2\mathrm{Tr}\left(h_2^2 - h_1^2\right)\right) \\ &+ \frac{1}{48}\epsilon^3\left[8\mathrm{Tr}(h_1^3 - h_2^3) + \mathrm{Tr}(h_1 - h_2)\left(6\mathrm{Tr}(h_2^2 - h_1^2) + (\mathrm{Tr}(h_1 - h_2))^2\right)\right].\end{aligned} \tag{23}$$

As a result the ϵ^2-part of the interaction in the above action is of the form (we omit higher order corrections here, as the formula gets too complicated and not transparent):

$$\begin{aligned}
S_{\text{int}}^{(2)} \sim & (\beta_0 + 4\beta_1 + 6\beta_2 + 4\beta_3 + \beta_4) \\
& + \frac{1}{2}\epsilon(\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4)\text{Tr}(h_1) \\
& + \frac{1}{2}\epsilon(\beta_0 + 3\beta_1 + 3\beta_2 + \beta_3)\text{Tr}(h_2) \\
& - \frac{1}{8}\epsilon^2 \left(\beta_3 + 4\beta_2 + 5\beta_1 + 2\beta_0\right) \text{Tr}(h_2^2) \\
& - \frac{1}{8}\epsilon^2 \left(\beta_1 + 4\beta_2 + 5\beta_3 + 2\beta_4\right) \text{Tr}(h_1^2) \\
& - \frac{1}{8}\epsilon^2 \left(2\beta_1 + 4\beta_2 + 2\beta_3\right) \text{Tr}(h_1 h_2) \\
& + \frac{1}{8}\epsilon^2 \left(\beta_2 + 2\beta_3 + \beta_4\right) \left(\text{Tr}(h_1)\right)^2 \\
& + \frac{1}{8}\epsilon^2 \left(\beta_0 + 2\beta_1 + \beta_2\right) \left(\text{Tr}(h_1)\right)^2 \\
& + \frac{1}{4}\epsilon^2 \left(\beta_1 + 2\beta_2 + \beta_3\right) \left(\text{Tr}(h_1)\text{Tr}(h_1)\right) \\
& + \epsilon^3 \cdots
\end{aligned} \tag{24}$$

Comparing the above expression with Sec. 12 we see that there are two quadratic terms in Sec. 24 that have the same coefficient proportional to $\beta_1 + 2\beta_2 + \beta_3$ but one of them vanishes in Sec. 12 and the other does not. As a conclusion, even the perturbative form of the spectral action for the interacting geometries cannot be equivalent to the usually assumed model of action for bimetric gravity.

Similarly, one can demonstrate that a second natural choice to identify h_1 and h_2 from the bimetric perturbative expansion with W_- and W_+ fields also leads to contradiction already in the second order of the expansion in ϵ.

5 The interaction in terms of invariants

Even though the usually assumed form of the action for the bimetric gravity is not compatible with the spectral interactions between geometries one has to observe that the assumed form of the action is very restrictive as it uses only the coefficients of the invariant polynomial of the matrix $\mathbb{X}$. This particular choice is quite elegant, yet it restricts a lot the possible interaction terms.

A natural question is, what is the invariant form of the interaction in the perturbative expansion, which is expressed as polynomials in the invariants of the matrix $\mathbb{X}$. As there are only four independent invariants, we assume that the perturbative action is polynomial of order at most 4 in $\mathbb{X}$:

$$\begin{aligned}
S_{\text{int}} \sim \int d^4x \sqrt{\det g_2} \Big(& \alpha_0 + \alpha_1 \text{Tr}'(\mathbb{X}) + \alpha_2 \text{Tr}'(\mathbb{X}^2) + \alpha_3 \left(\text{Tr}'(\mathbb{X})\right)^2 + \alpha_4 \text{Tr}'(\mathbb{X}^3) \\
& + \alpha_5 \text{Tr}'(\mathbb{X})\text{Tr}'(\mathbb{X}^2) + \alpha_6 \left(\text{Tr}'(\mathbb{X})\right)^3 + \alpha_7 \text{Tr}'(\mathbb{X}^4) + \alpha_8 \text{Tr}'(\mathbb{X})\text{Tr}'(\mathbb{X}^3) \\
& + \alpha_9 \left(\text{Tr}'(\mathbb{X}^2)\right)^2 + \alpha_{10} \text{Tr}'(\mathbb{X}^2) \left(\text{Tr}'(\mathbb{X})\right)^2 + \alpha_{11} \left(\text{Tr}'(\mathbb{X})\right)^4 \Big),
\end{aligned} \tag{25}$$

where for convenience we use $\text{Tr}' = \text{Tr} - 4$. The only term, which we did not expand earlier is the last one, $\text{Tr}(\mathbb{X}^4)$, and its expansion up to third order in ϵ, is

$$\begin{aligned}
\text{Tr}(\mathbb{X}^4) = & 4 + 2\epsilon \text{Tr}(h_1 - h_2) + \epsilon^2 \left(\text{Tr}(h_1^2) + 3\text{Tr}(h_2^2) - 4\text{Tr}(h_1 h_2)\right) \\
& + \epsilon^3 \left(-4\text{Tr}(h_2^3) + 6\text{Tr}(h_1 h_2^2) - 2\text{Tr}(h_1^2 h_2)\right).
\end{aligned} \tag{26}$$

We expand (25) in ϵ and compare it (up to order ϵ^3) with (12). This leads to a linear system of equations that has a four-parameter family of solutions, with

$$\begin{aligned}
\alpha_1 &= -16 + \alpha_4, & \alpha_2 &= 10 - \frac{3}{2}\alpha_4, \\
\alpha_3 &= -2 - \alpha_5, & \alpha_7 &= -1 - \frac{1}{4}\alpha_4, \\
\alpha_9 &= \frac{1}{2} - \frac{1}{4}\alpha_5 - \frac{3}{4}\alpha_8, & \alpha_{10} &= -\frac{1}{2}\alpha_6,
\end{aligned} \tag{27}$$

and $\alpha_0 = 0$.

It is, in particular, possible to find a unique solution in the form of a polynomial of the lowest order in $\mathbb{X}$, which is a polynomial of third order, an the resulting action reads,

$$S_{\text{int}} \sim \int d^4x \sqrt{\det g_2} \Big(-10\mathrm{Tr}'(\mathbb{X}) + 8\mathrm{Tr}'(\mathbb{X}^2) - 2\left(\mathrm{Tr}'(\mathbb{X})\right)^2 - 2\mathrm{Tr}'(\mathbb{X}^3) + \mathrm{Tr}'(\mathbb{X})\mathrm{Tr}'(\mathbb{X}^2) \Big). \tag{28}$$

In other words, in the third order in ϵ we can eliminate all the terms of order higher than three in $\mathbb{X}$. Passing back to traces the formula is even simpler,

$$S_{\text{int}} \sim \int d^4x \sqrt{\det g_2} \Big(2\mathrm{Tr}(\mathbb{X}) + 4\mathrm{Tr}(\mathbb{X}^2) - 2\left(\mathrm{Tr}(\mathbb{X})\right)^2 - 2\mathrm{Tr}(\mathbb{X}^3) + \mathrm{Tr}(\mathbb{X})\mathrm{Tr}(\mathbb{X}^2) \Big). \tag{29}$$

6 Conclusions and outlook

Having derived an explicit perturbative form of the *interaction* term between geometries using the spectral methods for a simple two-sheeted non-product geometry we find that although it resembles the bimetric gravity theory the coefficients of the interaction potential cannot be matched to such a model.

There are, however, many interesting features of our result. First of all, it confirms (perturbatively up to the third-order) that the nonlinear interaction term between the geometries is expressed through a function of the invariants of the matrix $\mathbb{X} = \sqrt{g_2^{-1} g_1}$. Although this is almost obvious, due to the general covariance of the spectral action, no explicit formula for this function is known. Here, we find its perturbative expansion around flat geometry, which can be used to study the stability of interacting geometries and cosmology models. This formulation opens also the possibility for further examination of the ghost problem in our model. At first, comparing the perturbative form of the action to Fierz-Pauli theory [18] one guesses that there will be ghosts as the action has only one quadratic term. Yet, the full analysis is more intricate and we postpone the detailed studies for the future.

Furthermore, the explicit form of perturbative terms is quadratic in the difference of the small perturbations, which indicates that the flat geometry is indeed stable. Moreover, only one of the linearized fields will be massive and interact with the massless linear perturbations.

The main result of the paper is that there exists a natural, canonical and *geometric* interaction between two adjacent geometries. Independently of the interpretation that relates it to brane interactions in the bulk, interacting universes, bimetric gravity or noncommutative geometry, the interaction is fixed in the same way the invariance fixes the usual action terms for gravity (the cosmological constant and the Einstein-Hilbert scalar curvature term). It is an open intriguing question of what are the physical consequences of such interactions between geometries and what effects they have on cosmology.

A Polynomial integrals over higher spheres

We review here technical tool of the computation, which are the integrals of polynomial functions over the unit spheres. We are interested in the value of the following quantity,

$$I_{n,m}^{\alpha_1\beta_1\ldots\alpha_m\beta_m} = \int_{\|x\|=1} d^n x\, x^{\alpha_1} x^{\beta_1} \ldots x^{\alpha_m} x^{\beta_m}, \tag{30}$$

i.e. the monomial integrals over a unit sphere. This can be done by the straightforward generalization into higher dimensional cases of the method presented in [19] for the 2-sphere (see also [20],[21]). By denoting

$$\gamma_j = \begin{cases} \alpha_k, & j = 2k-1 \\ \beta_k, & j = 2k \end{cases} \tag{31}$$

for $k = 1, \ldots, m$, we then have

$$I_{n,m}^{\alpha_1\beta_1\ldots\alpha_m\beta_m} \equiv I_{S_n}^{\gamma_1\ldots\gamma_{2m}} = \int_{\|x\|=1} d^n x x^{\gamma_1} \ldots x^{\gamma_{2m}}. \tag{32}$$

Let $S_n = \partial B_n \equiv S^{n-1}$ in $\mathbb{R}^n$. The following generalization of [19, Prop. 1], which can be proven by induction on m, holds:

Proposition A.1. *Let* $I_{B_n}^{\gamma_1\ldots\gamma_{2m}} = \int_{\|x\|\leq 1} d^n x x^{\gamma_1} \ldots x^{\gamma_{2m}}$. *Then*

$$I_{B_n}^{\gamma_1\ldots\gamma_{2m}} = \frac{1}{2m+n} I_{S_n}^{\gamma_1\ldots\gamma_{2m}}. \tag{33}$$

Similarly, [19, Prop. 2] can be easily generalized to arbitrary dimensions:

Proposition A.2.

$$I_{S_n}^{\gamma_1\ldots\gamma_{2m+2}} = \frac{1}{2m+n} \left[\delta^{\gamma_1\gamma_2} I_{S_n}^{\gamma_3\ldots\gamma_{2m+2}} + \ldots + \delta^{\gamma_1\gamma_{2m+2}} I_{S_n}^{\gamma_2\ldots\gamma_{2m+1}} \right]. \tag{34}$$

The proof is again based on the induction.

The explicit formulae used in this paper concern three values in the four-dimesional case, which we present explicitly,

$$I^{\gamma_1\ldots\gamma_{2m}} \equiv I_{S_4}^{\gamma_1\ldots\gamma_{2m}} = \int_{\|x\|=1} d^4 x x^{\gamma_1} \ldots x^{\gamma_{2m}}. \tag{35}$$

For $m = 0$ we have $I^0 = \mathrm{area}(S_4) = 2\pi^2$. Now, using Prop. A.2, we immediately get

$$I^{\gamma_1\gamma_2} = \frac{1}{4}\delta^{\gamma_1\gamma_2}\mathrm{area}(S_4) = \frac{\pi^2}{2}\delta^{\gamma_1\gamma_2} \tag{36}$$

$$I^{\gamma_1\gamma_2\gamma_3\gamma_4} = \frac{\pi^2}{12}\left[\delta^{\gamma_1\gamma_2}\delta^{\gamma_3\gamma_4} + \delta^{\gamma_1\gamma_3}\delta^{\gamma_2\gamma_4} + \delta^{\gamma_1\gamma_4}\delta^{\gamma_2\gamma_3}\right], \tag{37}$$

and

$$\begin{aligned} I^{\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6} = \frac{\pi^2}{96} \big[& \delta^{\gamma_1\gamma_2}\left(\delta^{\gamma_3\gamma_4}\delta^{\gamma_5\gamma_6} + \delta^{\gamma_3\gamma_5}\delta^{\gamma_4\gamma_6} + \delta^{\gamma_3\gamma_6}\delta^{\gamma_4\gamma_5}\right) + \\ & + \delta^{\gamma_1\gamma_3}\left(\delta^{\gamma_2\gamma_4}\delta^{\gamma_5\gamma_6} + \delta^{\gamma_2\gamma_5}\delta^{\gamma_4\gamma_6} + \delta^{\gamma_2\gamma_6}\delta^{\gamma_4\gamma_5}\right) + \\ & + \delta^{\gamma_1\gamma_4}\left(\delta^{\gamma_2\gamma_3}\delta^{\gamma_5\gamma_6} + \delta^{\gamma_2\gamma_5}\delta^{\gamma_3\gamma_6} + \delta^{\gamma_2\gamma_6}\delta^{\gamma_3\gamma_5}\right) + \\ & + \delta^{\gamma_1\gamma_5}\left(\delta^{\gamma_2\gamma_3}\delta^{\gamma_4\gamma_6} + \delta^{\gamma_2\gamma_4}\delta^{\gamma_3\gamma_6} + \delta^{\gamma_2\gamma_6}\delta^{\gamma_3\gamma_4}\right) + \\ & + \delta^{\gamma_1\gamma_6}\left(\delta^{\gamma_2\gamma_3}\delta^{\gamma_4\gamma_5} + \delta^{\gamma_2\gamma_4}\delta^{\gamma_3\gamma_5} + \delta^{\gamma_2\gamma_5}\delta^{\gamma_3\gamma_4}\right)\big]. \end{aligned} \tag{38}$$

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2.2.3 Towards generalized bimetric models

We proceed with the analysis of the doubled geometry models. The intriguing possibility of referring to them as modified bimetric theories which, in addition to having all the crucial features of bimetric gravity, are also derivable from purely geometric setup, is the main motivation for these studies.

In order to continue the discussion of two-sheeted models that are beyond the FLRW class, we begin with the ones in which the metrics are constant and diagonal. Even for this seemingly trivial case, the spectral action cannot be computed analytically in full glory. Therefore, we concentrate on the model in which both the metrics are of the form $\mathrm{diag}(b^2, b^2, a^2, a^2)$. Due to the convenient parametrization that occurs during the computations, we refer to this model as the Hopf model.

We compute the spectral action for such a model. The potential term can be parametrized in a way that highlights the presence of the features typical for bimetric models. It is now more complicated than in the case of the FLRW geometries, e.g. it contains logarithmic terms.

Moreover, the case with generic metrics is also partially discussed. We started an analysis that may allow for further numerical studies of the doubled geometry model in the full glory, that is, with arbitrary metrics.

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Towards modified bimetric theories within non-product spectral geometry

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We discuss class of doubled geometry models with diagonal metrics. Based on the analysis of known examples we formulate a hypothesis that supports treating them as modified bimetric gravity theories. Certain steps towards the generic case are then performed.

I. INTRODUCTION

The description of gravity in terms of geometric objects is the cornerstone of Einstein's General Relativity and leads to an intriguing possibility of geometrizing all of the fundamental interactions. One of the existing proposal is based on the noncommutative geometry [1] - a framework that puts on equal footing both the metric structure of manifolds, Yang-Mills-type theories and also the Higgs mechanism. The spectral description of manifolds [2] can be generalized into other than classical geometries like discrete spaces and their products with manifolds. The latter one leads to the definition of the so-called almost-commutative geometries that were successfully applied to the description of gauge theories [3–5]. Appropriate choice of the finite space allows e.g. for the formulation of the noncommutative Standard Model of Particle Physics. In this case the finite geometry is build on the matrix algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ whose choice is dictated by the gauge group of the model [9].

Yet another model of this type, but much more simpler, is the one studied by Connes and Lott [10], where the finite algebra is just $\mathbb{C} \oplus \mathbb{C}$ and corresponds to the two points. In this case, the product space can be thought of as $M \times \mathbb{Z}_2$, that is, we have two copies of the same manifold. One can further generalize this geometry and can allow for two distinct metrics on these two sheets [14]. Such a doubled geometry is beyond the usual almost-commutative framework and therefore is of a non-product type. Since the spectral action principle applied to a single copy produces the Hilbert-Einstein action, the natural question of the form of an action functional for this non-product type of geometries arises. The answer for generic choices of metrics is not known yet, but in the case of the Friedmann-Lemaître-Robertson-Walker (FRLW) type of Euclidean metrics this was done analitycally [13, 14], and the stability of certain solutions was also analysed [13]. It was demonstrated therein that the interaction between the metrics resembles features characteristic to bimetric gravity models [7, 8]. Despite numerous similarities, certain significant differences are also present. In particular, the interaction potential for bimetric model is a polynomial one, while for the two-sheeted model it is a rational function. Further similarities and differences for generic metrics were recently analysed in [19], where yet another interpretation of this model in terms of interacting branes was proposed.

In this note we discuss yet another class of models beyond the FLRW framework. We illustrate the generally claimed features on the simplified example - the so-called Hopf model. In this case the interaction potential has nontrivial logarithmic terms but it still possesses bimetric gravity characteristics. Finally, we make a general comment on the doubled geometry models which may allow for its future numerical studies.

II. THE GENERIC DIAGONAL MODEL

A framework of spectral geometry, allowing for an equivalent description of geometric objects in terms of algebraic data, originates from the observation that the geometry of a compact spin Riemannian manifold M can be encoded in the collection of data $(C^\infty(M), L^2(M), D_M)$ [2], where $L^2(M)$ is the Hilbert space of square-integrable spinors, and D_M is the Dirac operator, which can be written locally (with the use of the spin connection ω) as $i\gamma^\mu(\partial_\mu + \omega_\mu)$.

This system of data is a prerequisite for the notion of a spectral triple, a set (A, H, D) consisting of a unital $*$-algebra A represented in a faithful way on a Hilbert space H, on which the (possibly unbounded) densely defined self-adjoint operator D acts. In the generic case it is assumed that the commutators $[D, a]$, $a \in A$, are well-defined and can be (uniquely) extended to an element from $B(H)$, bounded operators on H. Furthermore, the resolvent of D has to be compact. Several further comptability conditions are imposed for certain applications [6, 9].

In addition to the aforementioned canonical spectral triple associated to a manifold M, the finite dimensional ones are well-understood [11, 12]. In this case both an algebra A_F and a Hilbert space H_F are finite dimensional, and D_F is just a matrix. One can go one step further and consider products of spectral triples considered so far. The almost-commutative geometry is a result of such a construction, where the first spectral triple in the product is the canonical one for a manifold M, and the other one is finite. In the case with $\dim M = 4$, the Dirac operator for the resulting triple is (pointwisely) $D_M \otimes 1 + \gamma_5 \otimes D_F$, where γ_5 is the usual grading in the Clifford algebra associated to the manifold M.

However, even for the product space with finite part being just the two points set, this is not the most general Dirac operator one can consider. Indeed, the operator

$$\mathcal{D} = \begin{pmatrix} D_1 & \gamma\Phi \\ \gamma\Phi^* & D_2 \end{pmatrix} \tag{II.1}$$

with a field Φ, which for our purposes is taken to be a constant, is an example of another candidate [14]. Here D_1, D_2 are two Dirac operators for M, but considered with two different Riemannian metrics g_1, g_2. The operator γ is a straightforward generalization of γ_5: $\gamma^* = \gamma$, $\{\gamma, \gamma^a\} = 0$ for all anti-Hermitian γ^a generating the Clifford algebra, $\{\gamma^a, \gamma^b\} = -2\delta^{ab}1$, but now $\gamma^2 = \kappa = \pm 1$ (instead of requiring $\kappa = 1$). These models are refered to as the doubled geometries [13].

We consider geometries of this type with the metric on each sheet chosen to be of the form

$$ds^2 = \sum_{j=0}^{3} a_j^2 \left(dx^j\right)^2, \tag{II.2}$$

where a_j, for $j = 0, 1, 2, 3$, are constants. The spin connection ω is identically zero since for the coframe $\{\theta^a\}$ we have $d\theta^a = 0$ for every $a = 0, \ldots, 3$, and the resulting Dirac operator therefore reads, $D = \sum_{j=0}^{3} \frac{1}{a_j}\gamma^j\partial_j$. The corresponding doubled geometry constructed out of these two sheets is therefore described by a Dirac operator of the form

$$\mathcal{D} = \sum_{j=0}^{3} A_j\gamma^j\partial_j + \gamma F, \tag{II.3}$$

where

$$A_j = \begin{pmatrix} \frac{1}{a_{1,j}} & \\ & \frac{1}{a_{2,j}} \end{pmatrix}, \quad F = \begin{pmatrix} & \Phi \\ \Phi^* & \end{pmatrix}. \tag{II.4}$$

The associated Laplace operator is hence given by

$$\mathcal{D}^2 = -\sum_{j=0}^{3} A_j^2 \partial_j^2 + \sum_{j=0}^{3} [F, A_j]\gamma\gamma^j \partial_j + \kappa F^2, \tag{II.5}$$

and one can then easily read the decomposition of its symbol into the homogeneous parts, $\sigma_{\mathcal{D}^2} = \mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{a}_2$.

Since our first goal is to determine the leading terms of the spectral action,

$$\begin{aligned} \mathcal{S}(\mathcal{D}) &= \Lambda^4 \operatorname{Wres}(\mathcal{D}^{-4}) + c\Lambda^2 \operatorname{Wres}(\mathcal{D}^{-2}) \\ &= \int_M \int_{\|\xi\|=1} \left(\Lambda^4 \operatorname{Tr}\operatorname{Tr}_{Cl} \mathfrak{b}_0^2 + c\Lambda^2 \operatorname{Tr}\operatorname{Tr}_{Cl} \mathfrak{b}_2\right), \end{aligned} \tag{II.6}$$

we have to find the symbol of the inverse of the Laplace operator, $\sigma_{\mathcal{D}^{-2}} = \mathfrak{b}_0 + \mathfrak{b}_1 + \mathfrak{b}_2 + ...$, what can be achieved by using the standard methods of pseudodifferential calculus [15]. (In the above equation Tr_{Cl} denotes the trace performed over the Clifford algebra and Tr is the usual matrix trace over two-by-two matrices.)

In our case we get

$$\mathfrak{b}_0 = \left(\sum_{j=0}^{3} A_j^2 \xi_j^2\right)^{-1}, \qquad \mathfrak{b}_2 = \mathfrak{b}_0\mathfrak{a}_1\mathfrak{b}_0\mathfrak{a}_1\mathfrak{b}_0 - \mathfrak{b}_0\mathfrak{a}_0\mathfrak{b}_0, \tag{II.7}$$

so that

$$\operatorname{Tr}_{Cl}(\mathfrak{b}_2) = -4\kappa\mathfrak{b}_0 \left(\sum_{j=0}^{3} [F, A_j]\mathfrak{b}_0[F, A_j]\xi_j^2 + F^2\right)\mathfrak{b}_0. \tag{II.8}$$

The only nonzero elements of the matrix $\mathfrak{b}_0$ are on its diagonal and they are equal to

$$(\mathfrak{b}_0)^i{}_i = \frac{1}{\sum_{j=0}^{3} A_{i,j}^2 \xi_j^2}, \tag{II.9}$$

where $A_{i,j} \equiv (A_j)^i{}_i = \frac{1}{a_{i,j}}$, and as a result of a straightforward computation we get

$$\begin{aligned} &\operatorname{Tr}\left(-4\kappa\mathfrak{b}_0 \sum_{j=0}^{3} [F, A_j]\mathfrak{b}_0[F, A_j]\mathfrak{b}_0\xi_j^2\right) \\ &= 4\kappa|\Phi|^2 \sum_{j,k=0}^{3} \frac{(A_{2,j} - A_{1,j})^2 (A_{1,k}^2 + A_{2,k}^2)}{\left(\sum_{l=0}^{3} A_{1,l}^2 \xi_l^2\right)^2 \left(\sum_{l=0}^{3} A_{2,l}^2 \xi_l^2\right)^2} \xi_j^2 \xi_k^2. \end{aligned} \tag{II.10}$$

The resulting spectral action is therefore of the form

$$\mathcal{S}(\mathcal{D}) \sim \int_M \left(\Lambda_e^2 \mathcal{S}_{\Lambda_e} + \alpha \widehat{V}(g_1, g_2)\right), \tag{II.11}$$

with

$$\mathcal{S}_{\Lambda_e} = \int_{\|\xi\|=1} \left\{ \left(\sum_{j=0}^{3} A_{1,j}^2 \xi_j^2\right)^{-2} + \left(\sum_{j=0}^{3} A_{2,j}^2 \xi_j^2\right)^{-2} \right\} \tag{II.12}$$

and

$$\widehat{V}(g_1, g_2) = \sum_{j,k=0}^{3} (A_{2,j} - A_{1,j})^2 (A_{1,k}^2 + A_{2,k}^2) \int_{\|\xi\|=1} \frac{\xi_j^2 \xi_k^2}{\left(\sum_{l=0}^{3} A_{1,l}^2 \xi_l^2\right)^2 \left(\sum_{l=0}^{3} A_{2,l}^2 \xi_l^2\right)^2} \tag{II.13}$$

where we have already introduced effective parametrization,

$$\Lambda_e^2 = \frac{12}{c}(\Lambda^2 - c\kappa|\Phi|^2), \qquad \alpha = 12|\Phi|^2\kappa, \tag{II.14}$$

and ommited the irrelevant global multiplicative constant. Therefore, the problem of finding the potential term describing the interaction between the two diagonal metrics reduces to compute linear combination of the integrals of the form

$$\int_{\|\xi\|=1} \frac{\xi_j^2 \xi_k^2}{\left(\sum_{l=0}^{3} A_{1,l}^2 \xi_l^2\right)^2 \left(\sum_{l=0}^{3} A_{2,l}^2 \xi_l^2\right)^2}. \tag{II.15}$$

Moreover, from the Eqn. (II.13) it immediately follows that $\widehat{V}(g_1, g_2) = \widehat{V}(g_2, g_1)$, that is, $\widehat{V}$ is symmetric under the interchange $g_1 \leftrightarrow g_2$. We further conjecture that the potential term can be written as

$$\widehat{V}(g_1, g_2) = 2\pi^2 \mathbb{V}\left(\sqrt{g_2^{-1} g_1}\right) \sqrt{\det\, g_2} \tag{II.16}$$

for some function $\mathbb{V}$. By the symmetry of $\widehat{V}$, to prove this claim it is enough to show that the function $\mathbb{V}'(g_1, g_2) := \frac{\widehat{V}(g_1, g_2)}{2\pi^2 \sqrt{\det g_2}}$ depends only on the eigenvalues of $\sqrt{g_2^{-1} g_1}$. We illustrate this hypothesis on a simple nontrivial example - the Hopf model - discussed in the forthcoming section.

III. THE HOPF MODEL

We consider here models with diagonal metrics given by $g_{00} = g_{11} = b^2$ and $g_{22} = g_{33} = a^2$, for which then have

$$(\mathfrak{b}_0)^1{}_1 = \frac{a_1^2 b_1^2}{a_1^2(\xi_0^2 + \xi_1^2) + b_1(\xi_2^2 + \xi_3^2)}, \qquad (\mathfrak{b}_0)^2{}_2 = \frac{a_2^2 b_2^2}{a_2^2(\xi_0^2 + \xi_1^2) + b_2(\xi_2^2 + \xi_3^2)} \tag{III.1}$$

and

$$\widehat{V}(g_1,g_2)=\frac{(b_1-b_2)^2}{b_1^2b_1^2}\int_{\|\xi\|=1}(\xi_0^2+\xi_1^2)\det(\mathfrak{b}_0)\mathrm{Tr}(\mathfrak{b}_0) + \frac{(a_1-a_2)^2}{a_1^2a_1^2}\int_{\|\xi\|=1}(\xi_2^2+\xi_3^2)\det(\mathfrak{b}_0)\mathrm{Tr}(\mathfrak{b}_0). \tag{III.2}$$

In order to parametrize the three-sphere $\|\xi\|=1$ we use here the following Hopf-like coordinates which resemble the symmetry of the system:

$$\xi_0=\cos\theta\cos\varphi,\quad \xi_1=\cos\theta\sin\varphi,\quad \xi_2=\sin\theta\cos\psi,\quad \xi_3=\sin\theta\sin\psi. \tag{III.3}$$

The angle θ is taken from $\left[0,\frac{\pi}{2}\right]$, while $0\le\psi\le 2\pi$, and the surface element in these coordinates is then given by $dS=\cos\theta\sin\theta d\theta d\varphi d\psi$.

As a result, we get

$$\det(\mathfrak{b}_0)\mathrm{Tr}(\mathfrak{b}_0) =\frac{a_1^2a_2^2b_1^2b_2^2\left[a_1^2b_1^2(a_2^2\cos^2\theta+b_2^2\sin^2\theta)+a_2^2b_2^2(a_1^2\cos^2\theta+b_1^2\sin^2\theta)\right]}{\left[a_1^2\cos^2\varphi(a_1^2\cos^2\theta+b_1^2\sin^2\theta)\sin^2\varphi\right]^2\left[a_2^2\cos^2\varphi(a_2^2\cos^2\theta+b_2^2\sin^2\theta)\sin^2\varphi\right]^2}. \tag{III.4}$$

Let us introduce the following notation

$$I_{\mu,c}=\int_{\|\xi\|=1}\frac{\xi_\mu^2\cos^2\theta\,dS}{\left[a_1^2\cos^2\varphi(a_1^2\cos^2\theta+b_1^2\sin^2\theta)\sin^2\varphi\right]^2\left[a_2^2\cos^2\varphi(a_2^2\cos^2\theta+b_2^2\sin^2\theta)\sin^2\varphi\right]^2}, \tag{III.5}$$

and

$$I_{\mu,s}=\int_{\|\xi\|=1}\frac{\xi_\mu^2\sin^2\theta\,dS}{\left[a_1^2\cos^2\varphi(a_1^2\cos^2\theta+b_1^2\sin^2\theta)\sin^2\varphi\right]^2\left[a_2^2\cos^2\varphi(a_2^2\cos^2\theta+b_2^2\sin^2\theta)\sin^2\varphi\right]^2}, \tag{III.6}$$

and notice that

$$I_{0,c}=I_{1,c}=I_{2,c}=I_{3,c},\qquad I_{0,s}=I_{1,s},\qquad I_{2,s}=I_{3,s}, \tag{III.7}$$

so that

$$\widehat{V}(g_1,g_2)=2a_1^2a_2^2b_1^2b_2^2\left\{a_1^2a_2^2(b_1^2+b_2^2)\left[\frac{(b_1-b_2)^2}{b_1^2b_2^2}I_{0,c}+\frac{(a_1-a_2)^2}{a_1^2a_2^2}I_{0,s}\right] +b_1^2b_2^2(a_1^2+a_2^2)\left[\frac{(b_1-b_2)^2}{b_1^2b_2^2}I_{0,s}+\frac{(a_1-a_2)^2}{a_1^2a_2^2}I_{2,s}\right]\right\}. \tag{III.8}$$

In order to find the final form of the potential it remains to compute the integrals $I_{0,c}$, $I_{0,s}$ and $I_{2,s}$. The result reads,

$$\widehat{V}(g_1,g_2)=\frac{2\pi^2}{(a_2b_1-a_1b_2)(a_2b_1+a_1b_2)^2}\left(F(a_1,a_2,b_1,b_2)+G(a_1,a_2,b_1,b_2)\right), \tag{III.9}$$

where

$$F(a_1,a_2,b_1,b_2)=4a_1^2a_2^2b_1^2b_2^2(a_1-a_2)(b_1-b_2)\log\left(\frac{a_1b_2}{a_2b_1}\right), \tag{III.10}$$

and

$$G(a_1,a_2,b_1,b_2) = (a_2^2b_1^2 - a_1^2b_2^2)\left[a_1^2b_1^2a_2(b_1-2b_2) + a_2^2b_2^2a_1(b_2-2b_1) + a_1^3b_1^2b_2 + a_2^3b_2^2b_1\right]. \tag{III.11}$$

We observe that for $a_1 = a_2$ the potential reduces to $\widehat{V}(g_1,g_2) = 2\pi^2a_1^2(b_1-b_2)^2$, while for $b_1 = b_2$ we have $\widehat{V}(g_1,g_2) = 2\pi^2(a_1-a_2)^2b_1^2$.

Since the logarithm vanishes if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ it would be, in principle, interesting to consider the limit of $\widehat{V}(g_1,g_2)$ when b_1 tends to $b_2\frac{a_1}{a_2}$. The value of the function V is indeterminated in this case, but the limit may still exists. Indeed, as a result we get

$$\lim_{b_1\to b_2\frac{a_1}{a_2}} \widehat{V}(g_1,g_2) = \frac{b_2^2}{a_2^2}(a_1-a_2)^2(a_1^2+a_2^2). \tag{III.12}$$

Introducing the new variable $x = \frac{b_1}{b_2}$ and $y = \frac{a_1}{a_2}$ we can write

$$\widehat{V}(g_1,g_2) = 2\pi^2\mathbb{V}\left(\sqrt{g_2^{-1}g_1}\right)\sqrt{\det\ g_2}, \tag{III.13}$$

where

$$\mathbb{V}\left(\sqrt{g_2^{-1}g_1}\right) = \frac{4x^2y^2(x-1)(y-1)}{(x-y)(x+y)^2}\log\left(\frac{y}{x}\right) + x^2y^2 + 1 - 2xy\frac{xy+1}{x+y}. \tag{III.14}$$

We observe that

$$\mathbb{V}\left(\sqrt{g_2^{-1}g_1}\right)\sqrt{\det\ g_2} = \mathbb{V}\left(\sqrt{g_1^{-1}g_2}\right)\sqrt{\det\ g_1}, \tag{III.15}$$

what illustrates the hypothesis.

IV. COMMENT ON GENERIC METRICS

The so far examined examples of doubled geometries suggest that these models can be thought of as certain modifications of bimetric theories as the potential term possesses features characteristic to this type of modified gravity theories. Despite the fact that series of non-trivial examples are already analysed, the derivation of the action in the generic case is still an open problem. In the approach we are using the main chalenge is related with the computation of certain integrals of rational functions defined over higher spheres:

$$I = \int_{\|\xi\|=1} \frac{d^4\xi}{A_{\mu\nu}\xi^\mu\xi^\nu}, \tag{IV.1}$$

with smooth $A_{\mu\nu}$, which can be further written as $A_{\mu\nu} = \Omega(\delta_{\mu\nu} + \epsilon_{\mu\nu})$ with $\Omega \in \mathbb{R}$ and $\epsilon_{\mu\nu}$ being symmetric and traceless. In the formula above ξ^α is the αth coordinate of vector ξ.

We make here some comments on the analysis of doubled geometry models in case when the tensors $A_{\mu\nu}$ does not differ sufficiently from the diagonal ones. This is not identical to the situation where the metrics are small perturbation of the Euclidean ones - for the discussion of the latter we refer to [19].

Assuming that $\max_{\mu,\nu} \|\epsilon_{\mu\nu}\|_\infty$ is sufficiently small we can expand in these parameters and write

$$I = \frac{1}{\Omega} \sum_{m \geq 0} (-2)^m \epsilon_{\alpha_1\beta_1} \dots \epsilon_{\alpha_m\beta_m} I_{4,m}^{\alpha_1\beta_1\dots\alpha_m\beta_m}, \tag{IV.2}$$

where

$$I_{n,m}^{\alpha_1\beta_1\dots\alpha_m\beta_m} = \int_{\mathbb{S}^{n-1}} d^n\xi \xi^{\alpha_1}\xi^{\beta_1} \dots \xi^{\alpha_m}\xi^{\beta_m} \tag{IV.3}$$

are polynomial integrals over higher spheres which can be evaluated generalizing the methods from [18] - see also [19] for further discussion.

Denoting

$$\gamma_j = \begin{cases} \alpha_k, & j = 2k-1, \\ \beta_k, & j = 2k \end{cases} \tag{IV.4}$$

we define $I^{\gamma_1\cdots\gamma_{2m}} = I_{4,m}^{\alpha_1\beta_1\dots\alpha_m\beta_m}$, and let $\Delta^{\gamma_1\cdots\gamma_{2m}}$ be the sum of product of deltas in $I^{\gamma_1\cdots\gamma_{2m}}$, i.e.

$$I^{\gamma_1\cdots\gamma_{2m}} = c_m \Delta^{\gamma_1\cdots\gamma_{2m}} \tag{IV.5}$$

with some real number c_m, and $\Delta^{\cdot\cdot\cdot} = \sum \delta^{\cdot\cdot}...\delta^{\cdot\cdot}$ Since ϵ is traceless not all terms in

$$\epsilon_{\gamma_1\gamma_2}...\epsilon_{\gamma_{2m-1}\gamma_{2m}} I^{\gamma_1\cdots\gamma_{2m}} \tag{IV.6}$$

are nonzero. Let $\mathcal{N}_{2m}$ be the number of nonzero terms, and consider the following problem. Suppose the numbers $1, ..., 2m$ are given, and we would like to use them to fill in an $1 \times 2m$ array T, with a given subdivision into 1×2 subarrays $T = T_1|T_2|...|T_m$, as follows:

- In the first entry of T_1 we put 1,
- For every $j = 1, ..., m$, we have $T_j = \left[a_j|b_j\right]$ with $a_j < b_j$,
- For every $j = 1, ..., m$, $a_j < a_{j+1}$.

Then $\mathcal{N}_{2m}$ is a number of such fillings for which there is no j such that T_j is of the form $\left[2l-1|2l\right]$, for some $l = 1, ..., m$.

Since, by symmetry of ϵ, any nonzero term in $\epsilon_{\gamma_1\gamma_2}...\epsilon_{\gamma_{2m-1}\gamma_{2m}} I^{\gamma_1\cdots\gamma_{2m}}$ produces $\mathrm{tr}(\epsilon^m)$, we have

$$\epsilon_{\gamma_1\gamma_2}...\epsilon_{\gamma_{2m-1}\gamma_{2m}} I^{\gamma_1\cdots\gamma_{2m}} = \mathcal{N}_{2m} c_m \mathrm{tr}(\epsilon^m), \tag{IV.7}$$

and the problem reduces to finding coefficients c_m. Since $\mathrm{area}(\mathbb{S}^3) = 2\pi^2$ we get $c_1 = \frac{\pi^2}{2}$. Moreover, by using the generalization of [18, Prop. 2] (see also [19, Prop. A.2]) one can easily find the recursive formula for c_m:

$$c_m = \frac{c_{m-1}}{4 + 2(m-1)}, \tag{IV.8}$$

and its solution reads

$$c_m = \frac{4\pi^2}{(2m+2)!!}. \tag{IV.9}$$

As a result,

$$I = \frac{1}{\Omega}\left[2\pi^2 + \frac{2\pi^2}{3}\mathrm{tr}(\epsilon^2) + 4\pi^2 \sum_{m\geq 3} \frac{(-2)^m}{(2m+2)!!}\mathcal{N}_{2m}\mathrm{tr}(\epsilon^m)\right]. \tag{IV.10}$$

In order to apply this result to a specific term of the action one has to first solve the combinatorial problem of finding the coefficients $\mathcal{N}_{2m}$, up to required order in m. We postpone for the future research the problem of determining set of metrics for which the rate of convergence of the above series is satisfactory for all the terms that appear in the action functional.

V. CONCLUSIONS AND OUTLOOK

The discussed doubled geometry model is an interesting possibility of going beyond the General Relativity. The explicit functional form of its action is derivable in the same way as the Hilbert-Einstein's one but with the use of a different geometry instead of the classical manifold. Here we extended the existing family of known examples for which the features characteristic to bimetric gravity models are present. We also made further steps towards the analysis of models that are beyond the class of such whose action is analytically computable. We remark that yet another approach based on a different type of noncommutativity can produce bimetric type of models [20]. It will be interesting to find some deeper relations between these two formulations - we postpone this for a future research.

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2.2.4 Outlook

The non-product geometry seems to be an intriguing possibility to go beyond the General Relativity within the noncommutative geometry framework. Considering a mild modification of the two-point Connes-Lott model we end up with a model that possesses the main features of the bimetric gravity. After performing partial analysis of the derived effective Lagrangian, we have demonstrated that this type of theory may be of physical interest and require further studies.

The analysis of the action under the assumption that both metrics differ infinitesimally from the Euclidean ones is the first step in analysing the possibility for the presence of certain types of ghosts in the doubled-geometry models. The preliminary results suggest that they could be found in these models, but since this problem is in general more subtle, further investigations are required. In particular, one may try to mimic certain arguments from e.g. [136, 145], where the issue of Boulware-Deser ghosts for bimetric gravity was discussed.

One can also further investigate the interpretation of the two parallel universes interacting in a nontrivial way. This also naturally suggests further generalization: models with finite (larger than two) copies of the same spacetime (or even different ones) with analogous Higgs-like interactions between every pair of sheets. One may speculate that this multi-sheeted universe could lead to interesting cosmological implications.

We have shown that certain non-product geometries can lead to intriguing cosmological models. The natural question that arises is what kind of non-product geometries can be used to derive other types of modified gravity theories. We hope that this path of research will be explored in the nearest future.

3 Conclusions and outlook

The paradigm of almost-commutative geometry was for years crucial for the *geometrization of physics*. We have demonstrated that by going beyond this framework one can find a geometric description for both the Standard Model of particle physics and also for cosmological theories that go beyond the General Relativity.

The formulation of the Standard Model as a non-product geometry allows for taking into account its Lorentzian structure and does not produce certain issues known from other descriptions. Several intriguing interplays between the Lorentzian symmetry, the chirality of the Dirac operator, the fermion doubling problem and the appearance of twisted structures requires further investigations. All of these aspects are present also in most of the existing attempts to solve the old fermion doubling problem. One can speculate that finding the precise, mathematically rigorous relation between the previous approaches and the one based on non-product geometry, may help with the understanding of the aforementioned interplays.

On the other hand, the application of non-product geometries for the description of modified gravity models gives an opportunity to study them from a geometric perspective. As an example, we have shown that simply modification of the two points Connes-Lott model results in the doubled geometry model that shares most of the features with bimetric gravity theories. Further analysis of these models are required and we expect that they may shed a new light on certain modified gravity theories.

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