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Report on the thesis
'Symmetry and Classification of Multipartite Entangled States'
by Adam Burchardt

General overview. The thesis entitled *'Symmetry and Classification of Multipartite Entangled States'* was prepared by Adam Burchardt under the supervision of prof. K. Życzkowski at the Jagiellonian University in Kraków. It is almost 100-pages long and is based on five articles co-authored by the Candidate. Three of them were already published in a good physical journal *Physical Review A,* whereas the other two were posted on the open-access preprint repository (arXiv.org), and most probably submitted to *Physical Review Letters*. For the reader's ease all these papers are attached at the end of the thesis. All of them were written with co-authors, however, it clearly follows from the thesis that Mr. Burchardt's contribution to their creation was substantial. Moreover, it should be pointed out here that two of these publications were prepared without the participation of his supervisor and in collaboration with younger colleagues from abroad. This suggests that the Candidate is open to collaboration but also evidences his scientific independence, which is certainly good.

The thesis is written in good English. Still, I managed to spot few linguistic errors here and there. For Author's use I append a sample of them at the end of this report.

The thesis consists of eight chapters, of which five contain original results (Chapters 2,3,5,6,7); each of these 'research' chapters corresponds to one of the publications the thesis is based on. These will be described in a more detailed way in the next part of this report. Here, let me shortly comment on the remaining three chapters (Chapters 1, 4 and 8). The first one contains a short introduction to the field, poses the problems that the Author aims to explore in the thesis, and outlines the content of the remaining chapters. Apart from this, Chapter 1 describes also a few basic notions of quantum theory as well as quantum information that are being used in the later parts of the thesis such as those of a qubit or a qutrit, entangled states, concurrence etc.

Chapter 4 is devoted to absolutely maximally entangled states or, more generally, to *k*-uniform states. It explains what role these states play in quantum information (for instance as codewords in quantum error correction), outlines their connections to certain combinatorial objects called orthogonal arrays, and presents examples thereof. This section does not introduce any new results; it is rather a prelude to the next chapters.

Chapter 8 concludes the thesis. On one hand, it summarizes all the results presented in it. On the other hand, it lists a number of related open problems that constitute a good inspiration for further research.

I have two comments regarding the structure of the thesis:

1.1. I think that the introductory part of the thesis is a bit too short and does not introduce all the notions and concepts that are later used in the thesis. Instead, these are presented in the other chapters, which created some repetitions and redundancies. For instance, the notion of orthogonal arrays is introduced twice: on page

25 and then, in a more detailed way on page 55. Also, perhaps it would be better to introduce the Latin and Graeco-Latin squares already in Chapter 4, not in Chapter 5.

1.2. Mr. Burchardt decided to prepare a full thesis and at the same time he mostly described his contribution to the obtained results. In my opinion the thesis should rather extend and enrich the content of publications and be a self-contained work, however it was sometimes easier for me to follow the argumentation presented in the publication rather than in the corresponding chapter. Moreover, this style of writing was not used consequently in the whole thesis as the last chapter is basically a copy of the Author's publication. In order to avoid this incoherent style of writing it would perhaps be better to present the thesis as a series of papers supplemented with a short introduction to the field and a summary of the obtained result.

Results. The research presented in the thesis is concerned with characterization of pure multipartite entangled states. Existence of entangled states is one of the most intriguing features of quantum theory. What is more, multipartite entanglement is a powerful resource for certain applications that have no classical analogue; just to mention quantum metrology or quantum cryptography. It is also vital for Bell non-locality which is another valuable resource in quantum information. At the same time, we have recently witnessed a huge progress in experimental implementation and controlled manipulation of various multipartite quantum states. All this indicates that characterization, detection, and classification of entanglement in multipartite systems is nowadays one of key problems in quantum information. While a considerable effort has already been devoted to studying this problem, our understanding of multipartite entanglement is still incomplete, even in the pure-state case. The thesis by Mr. Burchardt fits very well in this line of research, and, in fact, it provides a number of original and highly nontrivial results. Let me now briefly describe the results, chapter by chapter, and comment on them.

Chapter 2. In the second chapter, Mr. Burchardt introduced the notion of *m-resistant* states as those multipartite pure states that remain entangled after losing an arbitrary set of m subsystems, while become separable after losing yet another subsystem, and asks whether such states exist for any N. He provides two constructions of m-resistant states. The first one is based on permutationally invariant states and exploits their geometrical representation called Majorana representation, in which an N-qubit state can be represented by N points on a 3-dimensional sphere. As shown in the thesis, in the general case this approach allows one to construct N-qubit m-resistant states for three values of m. The second construction considers quantum states of higher local dimension than two and exploits their to mathematical object called orthogonal arrays, previously investigated for instance in [GZ14]. As proven in Proposition 3, using this relation it is possible to provide construction of m-resistant states for other values of m that are not covered by the first construction, yet at the cost of increasing the local dimension. The existence of m-resistance states for quantum systems consisting of N subsystems for a few values of N is nicely summarized in Table 2.1. It follows that while for most of pairs (N,m), m-resistant states exist, there are still some missing cases. The section ends with a short exploration of the case of large number of subsystems N, in which the resistance to particle loss of typical multipartite states can be inferred from the literature results.

While I find these constructions interesting, I have some comments that the Candidate might want to address during the defense.

1.1. While I appreciate that the Author makes a link between the problem of constructing m-resistant states in the bosonic case and the Majorana representation, I missed a more detailed explanation of how this representation is actually used in the construction and what is the intuition behind it.

1.2. I think it would be useful to at least shortly comment on how the resistance to noise of the obtained states would change if one considered a more physical situation in which they are mixed with white noise.

1.3. I also have two comments of more technical nature. First, it seems to me that for *m=0,* the Dicke state, as defined in Eq. (2.6) is rather |0...0>, not |1...1>. The second state is obtained for *m=N*, and therefore in Prop. 1, *m=0* should be replaced by *m=N,* or the definition in Eq. (2.6) should be modified accordingly. Also, there is 'k' instead of 'm' in Eq. (2.6). The second comment is that when introducing the definition of *m*-resistant states, the Author could have mentioned two other works that discuss similar notions: 1. H. J. Briegel, R. Raussendorf, Phys. Rev. Lett. **86**, 910 (2001) and 2. N. Brunner, T. Vértesi, Phys. Rev. A **86**, 042113 (2012). The first one appears in the list of references, but is cited only in Chapter 3.

Chapter 3. This chapter is devoted to characterization of multipartite states that exhibit certain symmetries. First, Mr. Burchardt proposes two methods to generate such states. The first one extends the construction of the well-known symmetric Dicke states, which are invariant under the action of the *N*-element permutation group, to arbitrary subgroups of it. The second one is based on the notion of hypergraphs, but it differs from the construction of hypergraph states known from the literature. It is more general than the first construction (in fact, it seems to reproduce the first one as a special case), and, importantly, in a nice way relates the symmetries of a given hypergraph to the symmetries of the corresponding state. Second, for some exemplary states such as the Dicke states, the Author analytically determines entanglement, as measured by concurrence, between any two subsystems as well as between a chosen subsystem and the rest. It is finally demonstrated how the excitation states can be prepared in physical systems; they are shown to be ground states of certain Hamiltonians with three-body interactions as well as the output states of certain quantum circuits. In the latter case, an exemplary state is simulated on the existing quantum computers.

3.1. It's nice that for certain highly symmetric graphs the Author was able to compute concurrence for all pairs of subsystems together with other related quantities. I missed, however, a more detailed discussion about what they actually tell us about the state or the corresponding graph; in particular, the Author could have added a few lines explaining what we learn from entanglement ratio defined in Eq. (3.14). Analogously, I think that a short intuitive explanation of why for some graphs concurrence between any two nodes vanishes expect for the particular case of nodes of distance two.

Chapter 5. This is in my opinion one of the most interesting parts of the thesis. It presents a very original solution to an open and nontrivial problem of existence of four-partite absolutely maximally entangled states of local dimension six. The question of existence of AME states in four-qu*d*it systems has already been resolved for any *d* except the particular case of *d=6* considered here. Interestingly, it cannot be solved by simply making use of the relation to Graeco-Latin squares as it is done for the other cases *d>2*. Here, the Author relates the problem to the quantum version of Graeco-Latin squares, in which pairs of 'classical' entries are replaced by two-qubit (in general entangled) states, and shows that finding the desired AME state can be approached by searching for a unitary matrix of dimension 6^2 which remains unitary after being partially transposed and reshuffled. The desired unitary matrix is then found by employing certain iterative numerical technique that is applied to a carefully chosen initial unitary matrices. Interestingly, this solves also the quantum version of the famous Euler's problem, which in the classical case is known not to have solution.

I highly value this part of the thesis it not only solves an interesting open problem in classification of the AME states, and thus significantly adds to our understanding of entanglement in multipartite systems, but also introduces a general and complete method that can be used to seek AME states in other quantum systems.

5.1. At the same time I have a small comment on the exposition of the results. I must admit that it was somehow easier for me to understand the problem and its solution from the corresponding manuscript attached to the thesis, rather than from the thesis itself. I think the Author concentrated mostly on his contribution to the solution of the problem which somehow influenced the clarity of the presentation. For instance, the quantum version of the Euler problem is not explicitly formulated in Chapter 5.

Chapter 6. The problem of equivalence of multipartite entangled states under SLOCC operations is a very important and intensively investigated problem. The main aim here is to tackle it for a particular class of *k*-uniform states, including also the absolutely maximally entangled states. As mentioned in the thesis, due to Kempf-Ness theorem, two *k*-uniform states are equivalent under SLOCC operations iff they are equivalent under local unitary operations, which narrows down the class of operations to consider. Here, Mr. Burchardt concentrates mostly on k-uniform states of minimal support, which are states with minimal number of terms in the decomposition in the computational basis.

First, it is demonstrated that in the case of *k*-uniform states such that $2k<N$ the set of local operations one needs to consider in order to check the equivalence of two states can further be narrowed down to a proper subset of unitary matrices, called monomial matrices; in fact, these are the only operations under which two *k*-uniform states can be equivalent. Using this observation, examples of AME states that are not equivalent under local operations are then provided.

Second, the Author moves on to the minimal-support AME states ($2k=N$), and determines another class of operations that is sufficiently large to cover local operations under which two AME states can be equivalent; these are based on the notion of Butson-type Hadamard matrices. The obtained results are them employed to explore equivalence under SLOCC operations of *k*-uniform states for a few lowest values of *k*. In particular, while by exploiting the link to orthogonal arrays it is conjectured that *2*-uniform *N*-qudit states of minimal support form a single SLOCC class (Conjecture 2), three-uniform states are proven to exhibit much more complex structure: in any composite system supporting *3*-uniform states of minimal support there exist infinitely many non-SLOCC-equivalent classes of such states. It should be pointed out that this chapter ends with a list of conjectures, which constitute excellent starting points for further research.

6.1. I think it would be useful to add some intuitive explanation of why in fact the Author considers states of minimal support. Or, in other words, why the obtained conclusions break down for states that do not meet this requirement.

6.2. I was really confused by the text that follows Remark 1. It defines d_min as the smallest dimension for which the inequality in Remark 1 is NOT satisfied, while a direct check reveals that all d_min listed in Table 6.1 actually satisfy this inequality. I guess this is only an editorial mistake and d_min is defined as the smallest d for which the inequality is satisfied. Also, 'Section 6.3' should be replaced by 'Table 6.1'.

Chapter 7. The last 'research' chapter of the thesis further explores the problems of SLOCC equivalence among multipartite quantum states. This time, however, the Author leaves the class of *k*-uniform states and considers arbitrary multiqubit states. Exploiting a certain type of polynomial entanglement measures that are invariant under the local action of the group SL(2,C), he proposes a general algorithm that allows one to verify whether two *N*-qubit states are equivalent under SLOCC operations; if these turn out to be equivalent, the algorithm returns also the corresponding SLOCC transformation. Importantly, the procedure is based only on a single measure and consists of a finite number of steps; thus, should always end with a solution, of course up to available computational resources. Next, using the three-tangle, the Candidate introduces a simple criterion (Proposition 13) enabling to check whether two states belonging to a certain family of four-qubit states, being linear combinations of four orthogonal GHZ-like states, are SLOCC equivalent, and based on this criterion provides a classification within this class.

I find this part of the thesis, next to Chapter 5, one of the most interesting and inspiring. It nicely combines various concepts such as SLOCC invariant entanglement measures, the Möbius transformation or the stereographic projection to provide a very general method to check SLOCC equivalence of multiqubit states based on a single entanglement measure. Still, there are two comments I would like to make.

7.1. While the algorithm needs a finite number of steps to output an answer, this number grows rapidly with the number of qubits. Consequently, it is unclear what is actually the maximal size of quantum system in which this method is applicable. I think it would have been useful if the Author commented on this issue in his thesis.

7.2. The method is designed for qubit states, and a natural question, at least to me, is how to generalize it to systems of higher local dimension.

Conclusion. This is in my opinion a good thesis. It presents a series of interesting and novel results, which significantly contribute to development of entanglement theory as well as quantum physics in general. Among them a few stand out as particularly interesting and inspiring: the solution of the quantum Euler problem together with the technique used to find it presented in Chapter 5 and the algorithm for verifying SLOCC equivalence of pure multiqubit quantum states discussed in Chapter 7. Another positive side of the thesis is the variety of quantum information tools and concepts used by Mr. Burchardt and the apparent ease with which the Author makes connections to notions from other areas of mathematical sciences, just to mention the stereographic projection, Möbius transformation, orthogonal Latin squares and others. On the other hand, the most important shortcomings of the thesis, expressed above, concern mostly the style of presentation. Nevertheless, they do not diminish my positive assessment of the scientific content of the thesis and the achievements of the PhD candidate.

Taking all this into account, I conclude that the thesis meets all the formal and customary requirements for a PhD thesis. Therefore, with no hesitation, I recommend awarding the doctoral degree to Adam Burchardt.

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Other comments.

1.1. Page 8: 'Having say that' → 'Having said that'.
1.2. Page 9: 'invertiable' → 'invertible'.
1.3. Page 11: 'If the discussion of entangled in ...' → 'If the discussion of entanglement in ...'.
1.4. Page 15: 'investigats' → 'investigates'.
1.5. Page 16: 'I discuss how THE presented solution opens THE door to A new area ...'.

2.1. Page 20: The sentence '... to a given m-resistant link to the associated quantum state ...' does not sound correct.
2.2. Page 21: 'invatigation' → 'investigation'.
2.3. Page 25: I think that in Proposition 2 it would be better to use 'corresponding' instead of 'relevant'.
2.4. Page 27: 'roves' → 'rows'.

3.1. Page 33: 'Clearly, that the group *H* ...' → 'Clearly, the group *H* ...'.
3.2. Page 34: I do not understand the meaning of 'are related with highly entanglement properties ...'.
3.3. Page 36: 'Dike-like' → 'Dicke-like'.
3.4. Page 35: I think that the cluster states are only a subset of all graph states.

4.1. Page 53: 'partise' → 'partite'
4.2. Page 54: I do not think it correct to use such expressions as 'larger entanglement properties' or 'maximizing entanglement properties'.
4.3. Page 54: There should be |S|=k instead of |bar{S}|=k in Def. 5.
4.4. Page 55: 'AME states do not exist for any numbers *N* and *d*' means that 'there are no AME states at all'. I think the Author wanted to say 'there exist pairs of *N* and *d* for which AME states do not exist'.
4.5. Page 55: 'Orthogonal array is a combinatorial arrangements' → 'An orthogonal array is a combinatorial arrangement'.
4.6. Page 55: It is more correct to say 'called of unit index'.

5.1. Page 62: The sentence 'Furthermore, such a square ...' does not sound correct.
5.2. Page 65: 'Furthermore, presented AME(4,6) ...' → 'Furthermore, the presented AME(4,6)'.

6.1. Page 74: 'LM-eoperations' → 'LM-operations'.
6.2. Page 74: I guess in '... root of unity, is ... ', '|AME(5,d)> state' should be replaced by 'an AME state'.
6.3. Page 74: 'with not necessary minimal support' → 'not necessarily with minimal support'.
6.4. Page 75: 'provid' → 'provide'.
6.5. Page 76: 'contains' → 'contain'. In fact, the whole sentence should be rephrased to 'Note that ..., the Butson matrices are classified and there are ... of them, respectively'.
6.6. Page 77: 'runs' → 'run'.
6.7. Page 76: It seems that Proposition 11 involves certain inequality stated only later in Remark 1. I think it should be stated already in Proposition 11.
6.8. Page 79: 'phases are equal' → 'phases equal'.
6.9. Page 80: 'summarizes' → 'summarized'.

7.1. Page 88: in the first statement of Theorem 2, I think the tensor power should be *N+1,* not *N.*
7.2. Page 89: 'will vanish for a roots ...' → 'will vanish for roots'.
7.3. Page 90: 'provides novel method' → 'provides a novel method'.
7.4. Page 92: 'convinient' → 'convenient'.
7.5. There is some inconsistency in notation between Eqs. (7.13) and (7.17). In Eq. (7.13) the cross-ratio is denoted by \lambda(z1,z2,z3,z4), whereas in Eq. (7.17) the lambda is dropped.

7.6. Page 94: I was confused by the sentence ‘Note that any generic 4-qubit state ...’. Precisely, the total Hilbert space’s dimension is here 16, whereas the subspace spanned by these states is four-dimensional, therefore I don’t why a generic four-qubit state should be of this form.
7.7. Page 95: In Eq. (7.19) the Author puts equality between the value of three-tangle and some pure states.