Monge-Ampère equation in hypercomplex geometry
By Marcin Sroka

This thesis studies problems related to the quaternionic Monge-Ampère equation. There are two main themes: 1. the Dirichlet problem for the quaternionic Monge-Ampère equation on domains in affine space $\mathbb{H}^n$, and 2. the quaternionic Monge-Ampère equation on compact manifolds which are hyperkähler with torsion (HKT).

The thesis opens with a very detailed and carefully written introduction to quaternionic geometry and the Moore determinant (Chapters 1–3), fixing his conventions carefully (these are not consistent throughout the literature), and giving plenty of intuition from the real and complex cases. I found these introductory chapters really pleasant to read, and was happy to finally find a place where all the "optimal" conventions are sorted out carefully.

Dirichlet problem for the quaternionic Monge-Ampère equation on domains in $\mathbb{H}^n$

In Chapter 4 the author goes on to define the quaternionic Monge-Ampère operator

$$MA_{\mathbb{H}}(u) = \det\left(\frac{\partial^2 u}{\partial \overline{q}_\alpha \partial q_\beta}\right),$$

for a real-valued C^2 function u on a domain in $\mathbb{H}^n$. In analogy with the complex case, he interprets this in terms of the top wedge product

$$(\partial\partial_J u)^n.$$

Functions for which the form $\partial\partial_J u$ is semipositive are then quaternionic plurisubharmonic (qpsh), and mimicking the approach of Bedford-Taylor one can define $MA_{\mathbb{H}}(u)$ for all u which are qpsh and locally bounded.

With this, one can then consider the Dirichlet problem as follows. Let $D \subset \mathbb{H}^n$ be a smooth bounded domain. One says that D is strictly quaternionic pseudoconvex if it admits a defining function which is strictly plurisubharmonic in the quaternionic sense. For such domains, the Dirichlet problem reads

$$\begin{cases} MA_{\mathbb{H}}(u) = f\mathrm{dVol} \quad \text{in } D, \\ u = \phi \quad \text{on } \partial D, \end{cases} \tag{1}$$

where $u \in C^0(\overline{D})$ is qpsh in D, $f \geq 0$ has some integrability properties and $\phi \in C^0(\partial D)$. Alesker and Harvey-Lawson had solved this Dirichlet problem when f is continuous. Later Wan extended this to $f \in L^q(D)$ for $q \geq 4$. The main result of Chapter 5 is that the Dirichlet problem can be solved with f merely in $L^q(D), q > 2$, and this threshold is sharp. Furthermore, he also shows that qpsh functions are locally in L^p for any $p < 2$, and again this threshold is sharp (and surprisingly, it is different from the integrability threshold of the fundamental solution of the quaternionic Monge-Ampère operator, which is $2n$). The proof of the solvability of the Dirichlet problem is ingenious and clean, and as mentioned it gives the optimal result in terms of the integrability of the RHS.

Furthermore, in Chapter 6 it is shown that if in addition $\phi \in C^{1,1}(\partial D)$ and f is bounded near ∂D, then the solution u is Hölder continuous, with explicit Hölder exponent.

Quaternionic Monge-Ampère equation on compact HKT manifolds

When working on compact manifolds (without boundary), there are several possible notions of what a "quaternionic" manifold should be, and the author does an excellent job at summarizing them. Arguably the most stringent one is the notion of a hyperkähler manifold, which is simply-connected and has a Riemannian metric which is Kähler with respect to a triple of complex structures that satisfy the quaternionic relations. It turns out that hyperkähler geometry is extremely rich, and has deep connections with algebraic geometry, but (up to complex deformations) very few examples are known.

A weaker notion is called HKT ("hyperkähler with torsion"), which originated in mathematical physics, and has also been much studied in differential geometry. And an even weaker notion is that of a hypercomplex manifold, which simply admits a triple of complex structures that satisfy the quaternionic relations.

A global version of the above quaternionic Monge-Ampère operator is then introduced on a general hypercomplex manifold in Chapter 7, which can be written as

$$(\Omega + \partial\partial_J u)^n,$$

where Ω is a "fundamental $(2,0)$-form" of a fixed hyperhermitian metric. The author then proposes a bold generalization of the Calabi-Yau theorem: on any compact hyperhermitian manifold, given any smooth function f there is a unique smooth function u (up to addition of a constant) and a unique $b \in \mathbb{R}$ such that $\Omega + \partial\partial_J u \geq 0$ and

$$(\Omega + \partial\partial_J u)^n = e^{f+b}\Omega^n.$$

When the metric is HKT with trivial canonical bundle, this conjecture was first proposed by Alesker-Verbitsky, and for general HKT manifolds this was posed by Alesker-Shelukhin. These authors devoted a lot of work towards proving this conjecture, but it was only solved in general by Alesker in the very restrictive case of flat hyperkähler manifolds. In general, the conjecture is known to be reduced to proving *a priori* $C^{2,\alpha}$ bounds for u, for some $\alpha > 0$. In 2017, Alesker-Shelukhin gave a very long and complicated proof of the a priori C^0 bound for u when the manifold is HKT. The main result of Chapter 8 is a new and much better proof of this result, which also allows for a weaker dependence on the RHS function f, by adapting some ideas of Cherrier and Tosatti-Weinkove from the complex case.

I should also remark that after this thesis was submitted, the author with S. Dinew succeeded in proving the above conjecture for all hyperkähler manifolds, using the above C^0 bound as a first step and deriving the required $C^{2,\alpha}$ bounds. This is a remarkable achievement.

Overall opinion

This thesis contains many beautiful original results, which show an impressive mastery of this difficult subject as well as great creative power. In particular, the thesis fulfills the requirement of "an original solution to the research problem, an original solution in scope of the application of one's research results in an economic or social domain, or an original artistic achievement." In fact, I believe this is one of the best theses in this field in the past few years, and I highly recommend the candidate for the degree of Doctor in Mathematics (PhD) **with distinction (cum laude).**

Some very minor corrections

- P.25, change "competitive convention" to "competing convention"
- P.36, change "in dept" to "in depth"
- Definition 4.2.1, is f here really $\mathbb{R}$-valued, or are $-\infty$-values also allowed, like in the complex case?
- P.78, change "spontaneous" to "sporadic"