Report on the Ph.D. thesis by M. Sroka "Monge-Ampère equation in hypercomplex geometry".

1 Real and complex Monge-Ampère equations are very classical and well studied. They play a central role in several areas of analysis and geometry. The Ph.D. thesis by M. Sroka deals with quaternionic versions of the Monge-Ampère (MA) equations. This class of PDE was introduced and studied for the first time by Alesker in 2003. Since than it was investigated by several people. Like real and complex MA-equations, it is a non-linear second order (possibly degenerate) elliptic equation. Many techniques developed for real and complex cases can be applied for the study of the quanternionic one after suitable modifications. At the same time it is already clear that new tools have to be developed for the quaternionic case.

In recent years there were other related attempts to generalize the real and complex Mongè-Ampere equations to other contexts, e.g. a series of works by Harvey and Lawson.

2 We denote by $\mathbb{H}^n$ the space of n-tuples of quaternions. We denote by $MA_{\mathbb{H}}(u)$ the quaternionic MA-operator on $\mathbb{H}^n$ (see the thesis for the definition).

The first main result of the thesis is Theorem 5.3.1 saying the following.

0.1 Theorem. *Let $D \subset \mathbb{H}^n$ be a bounded quanternionic strictly pseudoconvex domain. Let $p > 2$. Let $f \in L^p(D)$ be a non-negative function. Let $\phi \in C(\partial D)$. Then there exists unique function $u \in C(\bar{D})$ which is quaternionic plurisubharmonic in D and such that*

$$MA_{\mathbb{H}}(u) = f,$$
$$u|_{\partial D} = \phi.$$

This is the Dirichlet problem for quaternionic MA-equation. It was introduced and solved by Alesker in 2003 under notably stronger regularity assumptions on the initial data. These assumptions were relaxed by Zhu (2017) and Wan (2020). To the best of my knowledge Theorem 0.1 by Sroka is the strongest known result on the Dirichlet problem for quaternionic MA-equation. It is also shown that the condition $p > 2$ is optimal for existence of a continuous solution. The method of the proof is new and interesting. Notably, it uses results on the complex MA-equation.

Theorem 0.1 seems to me very interesting, both its statement and the method of the proof. I think it is very important for quaternionic MA-equations and pluripotential theory.

3 The second main result of the thesis, Theorem 6.5 (joint with Kolodziej), is as follows.

0.2 Theorem. *Under the assumptions of Theorem 0.1 assume in addition that $\phi \in C^{1,1}(\partial D)$. Then the only solution u belongs to $C^{0,\alpha}(\bar{D})$ for some $\alpha > 0$.*

This result is deep and very interesting. The method is novel for the field.

4 The third main result of the thesis belongs to another closely related subdirection in the theory of quaternionic MA-equations. This is a quaternionic version of the Calabi-Yau theorem. The complex Calabi-Yau theorem is a classical result in complex MA-equations with several applications in Kähler geometry. Its quaternionic version was introduced as a conjecture by Alesker and Verbitsky in 2010. So far there were obtained only partial results in this direction. All the known approaches are based on the continuity method and a priori estimates. One of the needed estimates is C^0-estimate of a solution. It was proved by Alesker and Shelukhin in 2017 in the needed generality. However their proof was rather lengthy and involved. Theorem 8.1 in the thesis presents a stronger C^0-estimate with a simpler proof. This result seems to me interesting and highly non-trivial.

5 To summarize, the main results of the thesis are very interesting, original, deep, and important for the field. *I strongly recommend Marcin Sroka for the Ph.D. title with the distinction (cum laude) provided he makes the following corrections.*

6 List of corrections.

- p.7, l.-10: add $f \geq 0$.
- p.36, last paragraph: strictly speaking [HL09c] studies the case of homogeneous MA-equations only. In particular its results do not completely cover those of Alesker who studied also non-homogeneous equations.
- p. 56, formula (II.5.1): add $f \geq 0$.
- p. 69, Theorem 5.3.1: again add $f \geq 0$.
- p.79, Theorem 7.1.9: my impression is that no simply connectedness of M is necessary. I am wondering why this assumption has been added.

Алескер

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